INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

Solving Linear and Nonlinear Fuzzy Fractional Volterra-Fredholm Integro Differential Equations Using Shehu Adomian Decomposition Method

S Luvis Savla[1], R Gethsi Sharmila[2]*

1 Research Scholar, PG & Research Department of Mathematics, Bishop Heber College (affiliated to Bharathidasan University), Tiruchirapppalli-17, Tamil Nadu, India
2 Associate Professor, PG & Research Department of Mathematics, Bishop Heber College (affiliated to Bharathidasan University), Tiruchirapppalli-17, Tamil Nadu, India

Check for updates

OPEN ACCESS

Received: 20-01-2024
Accepted: 29-01-2024
Published: 20-03-2024

Citation: Savla SL, Sharmila RG (2024) Solving Linear and Nonlinear Fuzzy Fractional Volterra-Fredholm Integro Differential Equations Using Shehu Adomian Decomposition Method. Indian Journal of Science and Technology 17(12): 1222-1230. https://doi.org/10.17485/IJST/v17i12.156

* **Corresponding author**.

gethsisharmila@gmail.com

Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Abstract

Objectives: In applied sciences and engineering, fuzzy fractional differential equations (FFDEs) and fuzzy fractional integral equations (FFIEs) are a crucial topic. The main objective of this work is to discover an analytical approximate solution for the fuzzy fractional Volterra-Fredholm integro differential equations (FFVFIDE). In the Caputo concept, fractional derivatives are regarded. **Methods:** The Shehu transform is challenging to exist for nonlinear problems. So, the Shehu transform is combined with the Adomian decomposition method is called the Shehu Adomian decomposition method (SHADM) and has been proposed to solve both linear and nonlinear FFVFIDEs. **Findings:** Both linear and nonlinear FFVIFIDEs can be solved using this technique. For nonlinear terms, Adomian polynomials have been used. The main benefit of this approach is that it converges quickly to the exact solution. Figures and numerical examples demonstrate the expertise of the suggested approach. **Novelty:** The comparison between the exact solution and numerical solution is shown in figures for various values of fractional order α. The numerical evolution demonstrates the efficiency and reliability of the proposed SHADM. The proposed approach is rapid, exact, and simple to apply and produce excellent outcomes.

Keywords: Fractional calculus; fuzzy number; Mittag Leffler function; Shehu Adomian decomposition method; fuzzy fractional Volterra-Fredholm integro differential equation

1 Introduction

Standard calculus is an extension of fractional calculus. This entails taking the derivative of functions during any order. With the help of fractional differential operators, memory and heredity features of various substances and procedures in electrical circuits, control, biology, electromagnetic processes, porous media, biomechanics, and electrochemistry have been recorded. A fuzzy number is used to represent measurement uncertainty.

Lotfi Zadeh created fuzzy sets in 1965, and since then, they have been used in a large number of applications. A new and significant topic of mathematics is the fuzzy theory of fractional and integro differential equations. In recent years, several academics have examined fuzzy fractional models and have produced innovative concepts for fuzzy fractional differential equations (FFDEs), fuzzy fractional integral equations (FFIEs), and fuzzy fractional integro-differential equations (FFIDEs). Then, FFDEs and FFIEs are used for a range of problems in applied mathematics, including those in biology, physics, medicine, and geography. In Physics, computer science, industrial engineering, artificial intelligence, and operations research, it has several practical problems. It may be translated to fractional order uncertain process problems.

Thete et al.[1] introduced the modified Laplace transform Adomian decomposition method to solve non-linear integro-differential equations. Abdul Majid et al.[2] solved the fuzzy Volterra- integro differential equations using the general linear method. Zia Ullah et al.[3] introduced the Laplace Adomian decomposition method to find the solution of fuzzy Volterra-integro differential equations. Georgieva et al.[4] used the Adomian decomposition method to solve nonlinear Volterra-Fredholm fuzzy integro-differential equations. Bargamadi et al.[5] proposed the second Chebyshev wavelets method to solve the system of fractional-order Volterra-Fredholm integro differential equations with weakly singular kernels.

In the field of Fuzzy Fractional Differential Equations (FFDEs) and Fuzzy Fractional Volterra-Fredholm Integro Differential Equations (FFVFIDE), there have been numerous implementations to find the exact and numerical solutions[6,7]. Nagwa et al.[8] proposed the usage of the fuzzy Adomian decomposition method for solving some fuzzy fractional partial differential equations. Alqudah et al.[9] introduced the novel numerical investigations of fuzzy Cauchy Reaction-Diffusion models via generalized fuzzy fractional derivative operators. In[10] the modified Adomian decomposition method (MADM) is used to determine the solution of the fuzzy fractional order Volterra-Fredholm integro-differential equations. In[11], the fractional residual power series (FRPS) technique is used to approximate the solutions for a class of fuzzy fractional Volterra integro-differential equations (FFVIDEs). Saima Rashid et al.[12] On fuzzy Volterra-Fredholm integro differential equation associated with Hilfer-generalized proportional fractional derivative [J]. Sachin Kumar et al.[13] proposed the Chebyshev spectral method for solving fuzzy fractional Fredholm-Volterra integro-differential equation. Gethsi Sharmila et al.[14] introduced the Shehu transform method for solving linear fuzzy fractional differential equations. Savla et al.[15] proposed the Shehu Adomian decomposition method for solving the fuzzy fractional biological population model.

However, the transform combined with the decomposition method is found to be the research gap in FFVFIDEs. So, the aforementioned articles assisted in the development of the SHADM for FFVFIDEs. In this paper, the Shehu transform method is merged with the Adomian Decomposition Method (ADM) to establish the methodology named the Shehu Adomian decomposition method (SHADM) is applied to solve fuzzy fractional Volterra-Fredholm integro differential equations. When compared to traditional methods, the proposed method can reduce the volume of computing effort while retaining high numerical accuracy; the size reduction equates to an enlargement in the approach's performance. In this paper, SHADM is utilized to generate both analytical and approximate solutions of linear and nonlinear FFVFIDE.

The general form of FFVFIDE:[7]

$$D^{\alpha}_{a^+}\tilde{y}(x;r) - p(x;r)\tilde{y}(x;r) = g(x;r) + \lambda \int_a^x k_1(s;r)\tilde{F}(y(s;r))ds + \gamma \int_a^b k_2(s;r)\tilde{G}(y(s;r))ds$$

with $q-1 < \alpha \leq q$, and subject to the initial condition

$$\tilde{y}^s(0) = [\underline{y}, \overline{y}] = \delta_i, \quad i = 0,1,\ldots,q-1, \quad q \in \mathrm{N}, \quad x > a \geq 0$$

Where λ, γ is a parameter. $D_t^{\alpha}\tilde{y}(x,t)$ is the fractional derivative of the function $\tilde{y}(x;r) = [\underline{y}(x;r), \overline{y}(x;r)]$ in Caputo's sense, $g(x;r)$ is the source term, $\widetilde{F}(y(t;r)) = [\underline{F}(y(t;r)), \overline{F}(y(t;r))]$, $\widetilde{G}(y(t;r)) = [\underline{G}(y(t;r)), \overline{G}(y(t;r))]$ may be the general linear or nonlinear differential operator. Also, $p(x;r)$ is a continuous real-valued function with non-negative or non-positive values on $[a,b], f : [a,b] \to R_F$ is continuous, and $\widetilde{y}^s(0) \in R_F$.

The construction of this paper is as follows: Section 1 affords the introduction of FFVIDEs and basic tools of fractional derivative, fuzzy number, and Shehu transform. Section 2 constructs the methodology of the Shehu Adomian decomposition method for FFVFIDE. Section 3 gives numerical examples of linear and nonlinear FFVFIDE to illustrate the accuracy of this method. Section 4 offers the conclusion.

1.1 Basic Tools

The essential definitions of fractional calculus, fuzzy numbers, and the Shehu transform are provided in this section.

1.1.1 Fractional Calculus

Definition 1.1

Fractional derivative in Caputo's sense is demarcated as:

$$ {}_{x}^{C}D_{b}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{x}^{b} (t-x)^{n-\alpha-1} \left(-\frac{d}{dt}\right)^{n} f(t) dt $$

Definition 1.2

The form of the Mittag-Leffler function is,

$$ E_{\sigma}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(n\sigma+1)},\ \sigma \in C,\ Re(\sigma) > 0 $$

A further generalization of the above equation is demarcated in the form

$$ E_{\sigma,\varphi}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(n\sigma+\varphi)},\ \sigma,\varphi \in C,\ Re(\sigma) > 0,\ Re(\varphi) > 0 $$

1.1.2 Fuzzy Number

Definition 1.3 (Triangular fuzzy number)

It's a three-point fuzzy number represented by $N = (\theta_1, \theta_2, \theta_3)$. The membership function of N is as follows:

$$ \mu_N(x) = \begin{cases} 0, & \psi < \theta_1, \\ \dfrac{\psi-\theta_1}{\theta_2-\theta_1}, & \theta_1 \le \psi \le \theta_2, \\ \dfrac{\theta_3-\psi}{\theta_3-\theta_2}, & \theta_2 \le \psi \le \theta_3, \\ 0, & \psi > \theta_3 \end{cases} \tag{1} $$

1.1.3 Shehu Transform

Definition 1.4

Over the set of functions, the Shehu transform of the function $f(t)$ is defined as [16]

$$ B = \left\{ f(t) | \exists,\ \tau_1, \tau_2 > 0,\ |f(t)| < M e^{\frac{|t|}{t_j}}\ if\ t \in (-1)^j \times (0,\infty) \right\} $$

By the following formula:

$$ f(u) = S[f(t)] = \int_{0}^{\infty} f(t) e^{-\frac{st}{u}} dt,\ t > 0 $$

Special properties of the Shehu transform are given:

1. $S[1] = \frac{u}{s}$;
2. $S\left[\frac{t^q}{\Gamma(q+1)}\right] = \frac{u^{q+1}}{s^{q+1}};\ q > 0$

Definition 1.5

The Caputo fractional derivative of Shehu transform is given by:

$$ S\left[D_t^{\alpha} f(x,t)\right] = \frac{s^{\alpha}}{u^{\alpha}} S[f(x,t)] - \sum_{k=0}^{q-1} \frac{s^{\alpha-(k+1)}}{u^{\alpha-(k+1)}} f^{(k)}\left(0^{+}\right), \quad q-1 < \alpha \le q. \tag{2} $$

2 Methodology

This section explains the methodology of Shehu Adomian decomposition method for FFVFIDEs.

2.1 Shehu Adomian Decomposition Method for Fuzzy Fractional Volterra-Fredholm Integro Differential Equations

In this section, the SHADM is derived to solve FFVFIDEs. To demonstrate the core principle of this method, consider a general FFVFIDE with an initial condition of the following kind:

$$D_t^\alpha \tilde{y}(x;r) = g(x;r) + p(x;r)\tilde{y}(x;r) + \lambda \int_0^x k_1(s;r)\tilde{F}(y(s;r))ds + \gamma \int_0^b k_2(s;r)\tilde{G}(y(s;r))ds \quad (3)$$

with $q-1 < \alpha \leq q$, and the initial condition

$$\tilde{y}^s(0) = [\underline{y}, \bar{y}] = \delta_i, \quad i = 0, 1, \ldots, q-1, \quad q \in \mathrm{N}$$

Where $D_t^\alpha \tilde{y}(x,t)$ is the fractional derivative of the function in Caputo's sense, $g(x,t)$ is the source term, $F(y(x))$ may be the general linear or nonlinear differential operator. The parametric form of (Equation (3)) is,

$$D_t^\alpha \underline{y}(x;r) = g(x;r) + p(x;r)\underline{y}(x;r) + \lambda \int_0^x k_1(s;r)\underline{F}(y(s;r)) + \gamma \int_0^b k_2(s;r)\underline{G}(y(s;r))ds \quad (4)$$

$$D_t^\alpha \bar{y}(x;r) = g(x;r) + p(x;r)\bar{y}(x;r) + \lambda \int_0^x k_1(s;r)\bar{F}(y(s;r)) + \gamma \int_0^b k_2(s;r)\bar{G}(y(s;r))ds \quad (5)$$

Using the Shehu transform on Equation (4),

$$S\left[D_t^\alpha \underline{y}(x;r)\right] = S[g(x;r)] + S[p(x;r)\underline{y}(x;r)] + S\left[\lambda \int_0^x k_1(s;r)\underline{F}(y(s;r))ds\right] + S\left[\gamma \int_0^b k_2(s;r)\underline{G}(y(s;r))ds\right]$$

Using the Shehu transform's differentiation characteristic and the initial conditions in Equation (3),

$$\begin{aligned} S[\underline{y}(x;r)] = \quad & \sum_{k=1}^{m-1} \frac{u^{-\alpha+k}}{s^{-\alpha+k}} \underline{y}^{(k)}(0;r) + \frac{u^\alpha}{s^\alpha} S[g(x;r)] + \frac{u^\alpha}{s^\alpha} S[p(x;r)\underline{y}(x;r)] \\ + \frac{u^\alpha}{s^\alpha} S & \left[\lambda \int_0^x k_1(s;r)\underline{F}(y(s;r))\right] ds + \frac{u^\alpha}{s^\alpha} S\left[\gamma \int_0^b k_2(s;r)\underline{G}(y(s;r))ds\right] \end{aligned} \quad (6)$$

Next, using the inverse Shehu transform's property, then

$$\begin{aligned} \underline{y}(x;r) = \quad & S^{-1}\left[\sum_{k=1}^{m-1} \frac{u^{-\alpha+k}}{s^{-\alpha+k}} \underline{y}^{(k)}(0;r)\right] + S^{-1}\left[\frac{u^\alpha}{s^\alpha} S[g(x;r)]\right] + S^{-1}\left[\frac{u^\alpha}{s^\alpha} S[p(x;r)\underline{y}(x;r)]\right] \\ + S^{-1} & \left[\frac{u^\alpha}{s^\alpha} S\left[\lambda \int_0^x k_1(s;r)\underline{F}(y(s;r))ds\right]\right] + S^{-1}\left[\frac{u^\alpha}{s^\alpha} S\left[\gamma \int_0^b k_2(s;r)\underline{G}(y(s;r))ds\right]\right] \end{aligned} \quad (7)$$

The SHADM represents the solution as,

$$\underline{y}(x) = \sum_{n=0}^{\infty} \underline{y}_n(x) \quad (8)$$

and decompose the nonlinear phrase as follows:

$$N\underline{y}(x) = \sum_{n=0}^{\infty} \underline{A}_n(y(x)) \quad (9)$$

for a sequence of Adomian's polynomials A_n are given by[17],

$$\underline{A}_n = \frac{1}{n!} \frac{d^n}{dp^n}\left[N\left(\sum_{i=0}^{\infty} \underline{y}_i\right)\right], n = 0, 1, 2, \ldots$$

For the nonlinear function $N\underline{y} = \underline{F}(y), M\underline{y} = \underline{G}(y)$, adomian polynomials,

$$\begin{aligned}
&\underline{A}_0 = \underline{F}(y_0)\\
&\underline{A}_1 = y_1\underline{F}^1(y_0)\\
&\underline{A}_2 = y_2\underline{F}^1(y_0) + \tfrac{1}{2!}y_1^2\underline{F}^2(y_0)\\
&\underline{A}_3 = y_3\underline{F}^1(y_0) + y_1y_2\underline{F}^2(y_0) + \tfrac{1}{3!}y_1^3\underline{F}^3(y_0)\\
&\vdots\\
&\underline{B}_0 = \underline{G}(y_0)\\
&\underline{B}_1 = y_1\underline{G}^1(y_0)\\
&\underline{B}_2 = y_2\underline{G}^1(y_0) + \tfrac{1}{2!}y_1^2\underline{G}^2(y_0)\\
&\underline{B}_3 = y_3\underline{G}^1(y_0) + y_1y_2\underline{G}^2(y_0) + \tfrac{1}{3!}y_1^3\underline{G}^3(y_0)\\
&\vdots
\end{aligned}$$

Substituting Equations (8) and (9) in Equation (7),

$$\begin{aligned}
\sum_{n=0}^{\infty}\underline{y}_n(x;r) =& S^{-1}\left[\sum_{k=0}^{m-1}\frac{u^{-\alpha+k+1}}{s^{-\alpha+k+1}}\underline{y}^{(k)}(0;r)\right] + S^{-1}\left[\frac{u^\alpha}{s^\alpha}S[g(x;r)]\right] + S^{-1}\left[\frac{u^\alpha}{s^\alpha}S\left[p(x;r)\sum_{n=0}^{\infty}\underline{y}_n(x;r)\right]\right]\\
&+S^{-1}\left[\frac{u^\alpha}{s^\alpha}S\left[\lambda\int_0^x k_1(s;r)\sum_{n=0}^{\infty}\underline{A}_n(y(s;r))ds\right]\right] + S^{-1}\left[\frac{u^\alpha}{s^\alpha}S\left[\gamma\int_0^b k_2(s;r)\sum_{n=0}^{\infty}\underline{B}_n(y(s;r))ds\right]\right]
\end{aligned} \tag{10}$$

If both sides of Equation (10) are described, as follows:

$$\begin{aligned}
\underline{y}_0(x;r) &= S^{-1}\left[\sum_{k=0}^{m-1}\tfrac{u^{-\alpha+k+1}}{s^{-\alpha+k+1}}\underline{y}^{(k)}(0;r)\right] + S^{-1}\left[\tfrac{u^\alpha}{s^\alpha}S[g(x;r)]\right]\\
\underline{y}_1(x;r) &= S^{-1}\left[\tfrac{u^\alpha}{s^\alpha}S[p(x;r)\underline{y}_0(x;r)\right] + S^{-1}\left[\tfrac{u^\alpha}{s^\alpha}S[\lambda\int_0^x k_1(s;r)\underline{A}_0(y(s;r))ds]\right]\\
&+S^{-1}\left[\tfrac{u^\alpha}{s^\alpha}S\left[\gamma\int_0^b k_2(s;r)\underline{B}_0(y(s;r))ds\right]\right]\\
\underline{y}_2(x;r) &= S^{-1}\left[\tfrac{u^\alpha}{s^\alpha}S[p(x;r)\underline{y}_1(x;r)\right] + S^{-1}\left[\tfrac{u^\alpha}{s^\alpha}S[\lambda\int_0^x k_1(s;r)\underline{A}_1(y(s;r))ds]\right]\\
&+S^{-1}\left[\tfrac{u^\alpha}{s^\alpha}S\left[\gamma\int_0^b k_2(s;r)\underline{B}_1(y(s;r))ds\right]\right]
\end{aligned}$$

The recursive relation is defined as follows:

$$\begin{aligned}
\underline{y}_0(x;r) &= S^{-1}\left[\sum_{k=0}^{m-1}\frac{u^{-\alpha+k+1}}{s^{-\alpha+k+1}}\underline{y}^{(k)}(0;r)\right] + S^{-1}\left[\frac{u^\alpha}{s^\alpha}S[g(x;r)]\right]\\
\underline{y}_{n+1}(x;r) &= S^{-1}\left[\frac{u^\alpha}{s^\alpha}S\left[p(x;r)\underline{y}_n(x;r)\right]\right] + S^{-1}\left[\frac{u^\alpha}{s^\alpha}S\left[\lambda\int_0^x k_1(s;r)\underline{A}_n(y(s;r))ds\right]\right]\\
&+S^{-1}\left[\frac{u^\alpha}{s^\alpha}S\left[\gamma\int_0^b k_2(s;r)\underline{B}_n(y(s;r))ds\right]\right]
\end{aligned}$$

Likewise, by doing the same calculation Equation (5) will be

$$\begin{aligned}
\sum_{n=0}^{\infty}\bar{y}_n(x;r) =& S^{-1}\left[\sum_{k=0}^{m-1}\frac{u^{-\alpha+k+1}}{s^{-\alpha+k+1}}\bar{y}^{(k)}(0;r)\right] + S^{-1}\left[\frac{u^\alpha}{s^\alpha}S[g(x;r)]\right] + S^{-1}\left[\frac{u^\alpha}{s^\alpha}S\left[p(x;r)\sum_{n=0}^{\infty}\bar{y}_n(x;r)\right]\right]\\
&+S^{-1}\left[\frac{u^\alpha}{s^\alpha}S\left[\lambda\int_0^x k_1(s;r)\sum_{n=0}^{\infty}\bar{A}_n(y(s;r))ds\right]\right] + S^{-1}\left[\frac{u^\alpha}{s^\alpha}S\left[\gamma\int_0^b k_2(s;r)\sum_{n=0}^{\infty}\bar{B}_n(y(s;r))ds\right]\right]
\end{aligned} \tag{11}$$

If both sides of Equation (11) are described:

$$\begin{aligned}
\overline{y}_0(x;r) &= S^{-1}\left[\sum_{k=0}^{m-1} \frac{u^{-\alpha+k+1}}{S^{-\alpha+k+1}}\overline{y}^{(k)}(0;r)\right] + S^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}S[g(x;r)]\right] \\
\overline{y}_1(x;r) &= S^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}S\left[p(x;r)\overline{y}_0(x;r)\right]\right] + S^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}S\left[\lambda \int_0^x k_1(s;r)\overline{A}_0(y(s;r))ds\right]\right] \\
&\quad + S^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}S\left[\gamma \int_0^b k_2(s;r)\overline{B}_0(y(s;r))ds\right]\right] \\
\overline{y}_2(x;r) &= S^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}S\left[p(x;r)\overline{y}_1(x;r)\right]\right] + S^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}S\left[\lambda \int_0^x k_1(s;r)\overline{A}_1(y(s;r))ds\right]\right] \\
&\quad + S^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}S\left[\gamma \int_0^b k_2(s;r)\overline{B}_1(y(s;r))ds\right]\right] \\
&\vdots
\end{aligned}$$

The recursive relation is defined as follows:

$$\begin{aligned}
\overline{y}_0(x;r) &= S^{-1}\left[\sum_{k=0}^{m-1} \frac{u^{-\alpha+k+1}}{s^{-\alpha+k+1}}\overline{y}^{(k)}(0;r)\right] + S^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}S[g(x;r)]\right] \\
\overline{y}_{n+1}(x;r) &= S^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}S\left[p(x;r)\overline{y}_n(x;r)\right]\right] + S^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}S\left[\lambda \int_0^x k_1(s;r)\overline{A}_n(y(s;r))ds\right]\right] \\
&\quad + S^{-1}\left[\frac{u^{\alpha}}{s^{\alpha}}S\left[\gamma \int_0^b k_2(s;r)\overline{B}_n(y(s;r))ds\right]\right]
\end{aligned}$$

Following the same procedure, the rest of the components $\left(\underline{y}_n(x;r), \overline{y}_n(x;r)\right)$ can be fully identified, and the series solutions may be fully determined. Then, using a truncated series, the analytical solution $(\underline{y}(x;r), \overline{y}(x;r))$ as follows:

$$\begin{aligned}
\underline{y}(x,t) &= \lim_{N\to\infty} \sum_{n=0}^{N} \underline{y}_n(x,t) \\
\overline{y}(x,t) &= \lim_{N\to\infty} \sum_{n=0}^{N} \overline{y}_n(x,t)
\end{aligned}$$

3 Results and discussion

In this section, some numerical examples of linear and nonlinear FFVFIDE using SHADM is presented.

Example 3.1

Consider the following Volterra type FFIDE[11]

$$D_{0^+}^{\alpha}\widetilde{y}(x;r) = [r-1, 1-r] + \int_0^x \widetilde{y}(s;r)ds,\ 0 < \beta \le 1,\ x \in [0,1] \tag{12}$$

with fuzzy initial condition $\widetilde{y}(0;r) = 0$.

The parametric form of Equation (12) is

$$D_{0^+}^{\alpha}\underline{y}(x;r) = (r-1) + \int_0^x \underline{y}(s;r)ds, \tag{13}$$

$$D_{0^+}^{\alpha}\overline{y}(x;r) = (1-r) + \int_0^x \overline{y}(s;r)ds \tag{14}$$

Applying SHADM in section 3 on Equation (13), then obtain

$$\begin{aligned}
\underline{y}_0(x;r) &= (r-1)\frac{x^{\alpha}}{\Gamma(\alpha+1)} \\
\underline{y}_1(x;r) &= (r-1)\frac{x^{(2\alpha+1)}}{\Gamma(2\alpha+2)} \\
\underline{y}_2(x;r) &= (r-1)\frac{x^{(3\alpha+2)}}{\Gamma(3\alpha+3)} \\
&\vdots \\
\underline{y}_n(x;r) &= (r-1)\frac{x^{((n+1)\alpha+n)}}{\Gamma((n+1)\alpha+n)} \\
&\vdots
\end{aligned}$$

Applying SHADM in section 3 on Equation (14), then obtain

$$\begin{gathered}\bar{y}_0(x;r) = (1-r)\frac{x^{\alpha}}{\Gamma(\alpha+1)}\\ \bar{y}_1(x;r) = (1-r)\frac{x^{(2\alpha+1)}}{\Gamma(2\alpha+2)}\\ \bar{y}_2(x;r) = (1-r)\frac{x^{(3\alpha+2)}}{\Gamma(3\alpha+3)}\\ \vdots\\ \bar{y}_n(x;r) = (1-r)\frac{x^{((n+1)\alpha+n)}}{\Gamma((n+1)\alpha+n)}\\ \vdots\end{gathered}$$

Continuing in this way,

$$\begin{aligned}\underline{y}(x;r) = & \lim_{N\to\infty}\sum_{n=0}^{N}\underline{y}_n(x;r)\\ & = (r-1)x^{\alpha}E_{\alpha+1,\alpha+1}\left(x^{\alpha+1}\right)\\ \bar{y}(x;r) & = \lim_{N\to\infty}\sum_{n=0}^{N}\bar{y}_n(x;r)\\ & = (1-r)x^{\alpha}E_{\alpha+1,\alpha+1}\left(x^{\alpha+1}\right)\end{aligned}$$

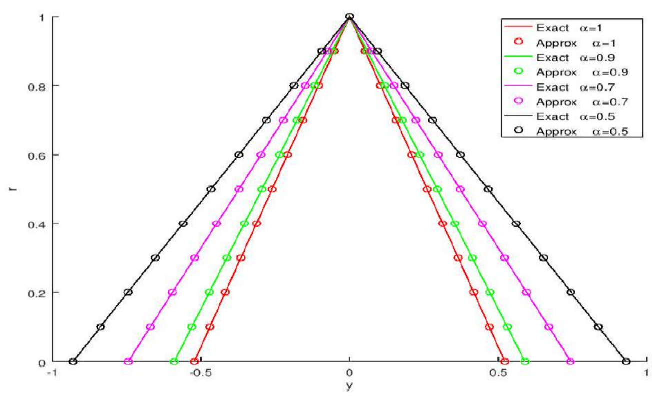

Fig 1. The approximate and exact solution of Equation (12) for various α.

Figure 1 represents the exact and approximate solution using SHADM for different fractional orders $\alpha = 0.5, 0.7, 0.9$, and the integer order 1. In this figure, the approximate solution almost coincides with the exact one.

Example 3.2

Consider the following nonlinear FFIDE of Fredholm type[7]

$$D_{0^+}^{\alpha}\widetilde{y}(x;r) = \widetilde{f}(x;r) + 0.1x^2\int_0^1 s\widetilde{y}^2(s;r)ds,\ 0<\beta\leq 1,\ x\in[0,1],\ \widetilde{y}(0;r)=0 \tag{15}$$

Where $[\widetilde{f}f(x;r)]^r = \left[rx - \frac{(rx)^2}{40}, (2-r) - \frac{((2-r)x)^2}{40}\right]$.

The parametric form of Equation (15) is

$$D_{0^+}^{\alpha}\underline{y}(x;r) = \underline{f}(x;r) + 0.1x^2\int_0^1 s\underline{y}^2(s;r)ds \tag{16}$$

$$D_{0^+}^{\alpha}\bar{y}(x;r) = \bar{f}(x;r) + 0.1x^2\int_0^1 s\bar{y}^2(s;r)ds \tag{17}$$

Applying SHADM in section 3 on Equation (16), then obtain

$$
\begin{aligned}
\underline{y}_0(x;r) &= \frac{x^{\alpha}}{\Gamma(\alpha+1)}\left(r-\frac{(rx)^2}{40}\right)\\
\underline{y}_1(x;r) &= \frac{x^{\beta+2}(0.1)r^2}{2\Gamma^3(\beta+1)}\left(\frac{1}{(\beta+1)}-\frac{r}{20(\beta+2)}+\frac{r^2}{40^2(\beta+3)}\right)\\
\underline{y}_2(x;r) &= \frac{x^{\beta+2}}{2\Gamma^5(\beta+1)}\frac{(0.1)^2r^2}{(\beta+2)}\left(\frac{1}{(\beta+1)}-\frac{r}{20(\beta+2)}+\frac{r^2}{40^2(\beta+3)}\right)\left(r-\frac{(rx)^2}{40}\right)\\
&\vdots
\end{aligned}
$$

Applying SHADM in section 3 on Equation (17), then obtain

$$
\begin{aligned}
\overline{y}_0(x;r) &= \frac{x^{\alpha}}{\Gamma(\alpha+1)}\left((2-r)-\frac{((2-r)x)^2}{40}\right)\\
\overline{y}_1(x;r) &= \frac{x^{\beta+2}(0.1)(2-r)^2}{2\Gamma^3(\beta+1)}\left(\frac{1}{(\beta+1)}-\frac{(2-r)}{20(\beta+2)}+\frac{(2-r)^2}{40^2(\beta+3)}\right)\\
\overline{y}_2(x;r) &= \frac{x^{\beta+2}}{2\Gamma^5(\beta+1)}\frac{(0.1)^2(2-r)^2}{(\beta+2)}\left(\frac{1}{(\beta+1)}-\frac{(2-r)}{20(\beta+2)}+\frac{(2-r)^2}{40^2(\beta+3)}\right)\left((2-r)-\frac{((2-r)x)^2}{40}\right)\\
&\vdots
\end{aligned}
$$

Then the approximate solution is,

$$
\begin{aligned}
\underline{y}(x;r) = {} & \frac{x^{\alpha}}{\Gamma(\alpha+1)}\left(r-\frac{(rx)^2}{40}\right)+\frac{x^{\beta+2}(0.1)r^2}{2\Gamma^3(\beta+1)}\left(\frac{1}{(\beta+1)}-\frac{r}{20(\beta+2)}+\frac{r^2}{40^2(\beta+3)}\right)\\
& +\frac{x^{\beta+2}}{2\Gamma^5(\beta+1)}\frac{(0.1)^2r^2}{(\beta+2)}\left(\frac{1}{(\beta+1)}-\frac{r}{20(\beta+2)}+\frac{r^2}{40^2(\beta+3)}\right)\left(r-\frac{(rx)^2}{40}\right)+\ldots.\\
\overline{y}(x;r) = {} & \frac{x^{\alpha}}{\Gamma(\alpha+1)}\left((2-r)-\frac{((2-r)x)^2}{40}\right)+\frac{x^{\beta+2}(0.1)(2-r)^2}{2\Gamma^3(\beta+1)}\left(\frac{1}{(\beta+1)}-\frac{(2-r)}{20(\beta+2)}+\frac{(2-r)^2}{40^2(\beta+3)}\right)\\
& +\frac{x^{\beta+2}}{2\Gamma^5(\beta+1)}\frac{(0.1)^2(2-r)^2}{(\beta+2)}\left(\frac{1}{(\beta+1)}-\frac{(2-r)}{20(\beta+2)}+\frac{(2-r)^2}{40^2(\beta+3)}\right)\left((2-r)-\frac{((2-r)x)^2}{40}\right)+\ldots
\end{aligned}
$$

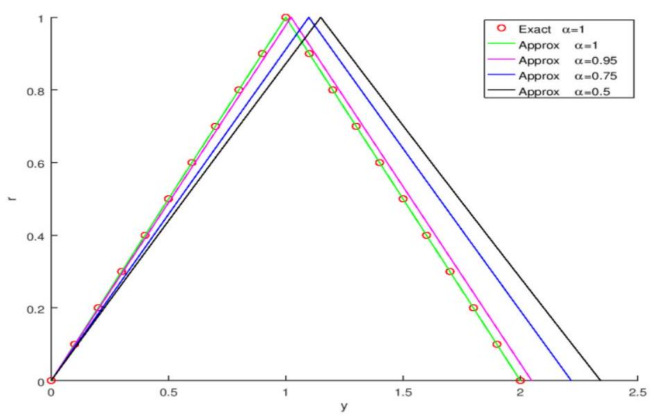

Fig 2. The approximate and exact solution of Equation (15) for various α

Figure 2 represents the approximate solution using SHADM for different fractional orders $\alpha = 0.5, 0.7, 0.9$, and also the exact and approximate solution is shown for the integer order 1. In this figure, the comparison is also given for the exact and approximate solution when the order is an integer.

4 Conclusion

This study is obtained an analytical and approximate solution for linear and nonlinear FFVFIDEs. The solution of the linear and nonlinear FFVFIDEs is accomplished using SHADM. Both weak and highly nonlinear problems can be solved using this technique. The method under consideration is distinctive in that it is based on Adomian polynomials, which enables speedy convergence of the discovered solution for the nonlinear term of the problem, and it employs a straightforward method to evaluate the solution for FFVFIDEs. When increasing the number of terms in the SHADM the approximate solution obtains its accuracy. From the figures, it concluded that the approximation value almost matches the exact answer when the derivative order approaches an integer. To investigate the behaviour, we have constructed two-dimensional graphs correlating to each illustrative example and showing it at different points of uncertainty and fractional orders α. It may be inferred that the suggested approach is rapid, exact, and simple to apply and produce excellent outcomes.

References

1) Thete RB, Jain A. Solution of non-linear integro-differential equations by using modified Laplace transform Adomian decomposition method. *Malaya Journal of Matematik*. 2021;09(01):1199–1203. Available from: https://www.malayajournal.org/index.php/mjm/article/view/1246.
2) Majid ZA, Rabiei F, Hamid FA, Ismail F. Fuzzy Volterra Integro-Differential Equations Using General Linear Method. *Symmetry*. 2019;11(3):1–18. Available from: https://doi.org/10.3390/sym11030381.
3) Ullah Z, Ahmad S, Ullah A, Akgül A. On solution of fuzzy Volterra integro-differential equations. *Arab Journal of Basic and Applied Sciences*. 2021;28(1):330–339. Available from: https://doi.org/10.1080/25765299.2021.1970874.
4) Georgieva A, Spasova M. Solving nonlinear Volterra-Fredholm fuzzy integro-differential equations by using Adomian decomposition method. In: APPLICATIONS OF MATHEMATICS IN ENGINEERING AND ECONOMICS (AMEE'20): Proceedings of the 46th International Conference "Applications of Mathematics in Engineering and Economics";vol. 2333, Issue 1 of AIP Conference Proceedings. AIP Publishing. 2021. Available from: https://doi.org/10.1063/5.0041602.
5) Bargamadi E, Torkzadeh L, Nouri K, Jajarmi A. Solving a System of Fractional-Order Volterra-Fredholm Integro-Differential Equations with Weakly Singular Kernels via the Second Chebyshev Wavelets Method. *Fractal and Fractional*. 2021;5(3):1–14. Available from: https://doi.org/10.3390/fractalfract5030070.
6) Chakraverty S, Tapaswini S, Behera D. Fuzzy Arbitrary Order System: Fuzzy Fractional Differential Equations and Applications. John Wiley & Sons. 2016. Available from: https://doi.org/10.1002/9781119004233.
7) Hasan S, Maayah B, Bushnaq S, Momani S. A modified reproducing Kernel Hilbert space method for solving fuzzy fractional integro-differential equations. *Boletim da Sociedade Paranaense de Matemática*. 2023;41:1–16. Available from: https://doi.org/10.5269/bspm.52289.
8) Saeed NA, and DBP. Usage of the Fuzzy Adomian Decomposition Method for Solving Some Fuzzy Fractional Partial Differential Equations. *Advances in Fuzzy Systems*. 2024;2024:1–15. Available from: https://doi.org/10.1155/2024/8794634.
9) Alqudah MA, Ashraf R, Rashid S, Singh J, Hammouch Z, Abdeljawad T. Novel Numerical Investigations of Fuzzy Cauchy Reaction–Diffusion Models via Generalized Fuzzy Fractional Derivative Operators. *Fractal and Fractional*. 2021;5(4):1–40. Available from: https://doi.org/10.3390/fractalfract5040151.
10) Ahmad N, Ullah A, Ullah A, Ahmad S, Shah K, Ahmad I. On analysis of the fuzzy fractional order Volterra-Fredholm integro-differential equation. *Alexandria Engineering Journal*. 2021;60(1):1827–1838. Available from: https://doi.org/10.1016/j.aej.2020.11.031.
11) Alaroud M, Al-Smadi M, Ahmad RR, Din UKS. An Analytical Numerical Method for Solving Fuzzy Fractional Volterra Integro-Differential Equations. *Symmetry*. 2019;11(2):1–19. Available from: https://doi.org/10.3390/sym11020205.
12) Rashid S, Jarad F, Abualnaja KM. On fuzzy Volterra-Fredholm integrodifferential equation associated with Hilfer-generalized proportional fractional derivative. *AIMS Mathematics*. 2021;6(10):10920–10946. Available from: https://doi.org/10.3934/math.2021635.
13) Kumar S, Nieto JJ, Ahmad B. Chebyshev spectral method for solving fuzzy fractional Fredholm–Volterra integro-differential equation. *Mathematics and Computers in Simulation*. 2022;192:501–513. Available from: https://doi.org/10.1016/j.matcom.2021.09.017.
14) Sharmila RG, Savla L. Shehu Transform Method for Linear Fuzzy Fractional Differential Equations. *Compliance Engineering*. 2024;12(1):257–267. Available from: https://www.researchgate.net/publication/377964543_Shehu_Transform_Method_for_Linear_Fuzzy_Fractional_Differential_Equations.
15) Savla SL, Sharmila RG. Solving Fuzzy Fractional Biological Population Model using Shehu Adomian Decomposition Method. *Indian Journal Of Science And Technology*. 2023;16(Special Issue 3):44–54. Available from: https://doi.org/10.17485/IJST/v16iSP3.icrtam151.
16) Maitama S, Zhao W. New Integral Transform: Shehu Transform a Generalization of Sumudu and Laplace Transform for Solving Differential Equations. *International Journal of Analysis and Applications*. 2019;17(2):167–190. Available from: https://doi.org/10.28924/2291-8639-17-2019-167.
17) Lydia SK. Numerical solution of Volterra integro differential equations of fractional order using Mahgoub Adomian decomposition method. *International Journal of Research in Advent Technology (IJRAT)*. 2019;(Special Issue):113–118. Available from: https://ijrat.org/downloads/Conference_Proceedings/ICAMBC-19/paper%20ID-25.pdf.