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RESEARCH ARTICLE

# Solving Time-Fractional Fitzhugh–Nagumo Equation using Homotopy Perturbation Method

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## Abstract

**Objectives:** This study aims to explore solutions to the time-fractional Fitzhugh-Nagumo equation, a nonlinear reaction-diffusion equation. **Method:** We utilize the Homotopy Perturbation Method (HPM) as a proficient analytical approach for addressing the time-fractional Fitzhugh-Nagumo equation. The HPM offers a structured method for deriving approximate solutions in the shape of convergent series, enabling accurate solutions even for intricate nonlinear fractional equations. **Finding:** The series solution obtained is validated by comparing it with numerical methods, showcasing its precision and effectiveness. Additionally, we assessed the error across various time and space values. Our analysis and computations reveal that the Homotopy Perturbation Method (HPM) stands out for providing precise approximations while maintaining computational efficiency. It's clear that this method presents a robust alternative to conventional numerical techniques, particularly in situations where analytical solutions are difficult to obtain. **Novelty:** The application of the Homotopy Perturbation Method to the Time-fractional Fitzhugh-Nagumo Equation has been effectively explored, with specific examples showing a strong agreement between the exact solution and the obtained solution.

**Keywords:** Time-Fractional Fitzhugh–Nagumo Equation; Homotopy Perturbation Method; Riemann-Liouville fractional integral; Caputo fractional derivative; Fractional Homotopy Perturbation Method

## 1 Introduction

In recent years, considerable attention has been directed towards the burgeoning field of "Fractional Calculus" by numerous researchers, driven by its diverse applications across various scientific and engineering domains [1–5]. Introducing fractional operators into classical differential equations results in complex challenges when solving resulting fractional-order partial differential equations [6]. As a consequence, researchers tend to favor numerical methods over analytical ones. The Fitzhugh-Nagumo (FN) equation stands as a pivotal nonlinear reaction-diffusion equation, presenting significant challenges due to its highly nonlinear nature [7]. This equation finds extensive utility in investigating biological systems characterized by excitability, such as neural communication facilitated by nerve cells through electrical signaling. Serving as a simplification of the Hodgkin-Huxley model, the FN equation aims to depict the membrane potential of a nerve axon. Consequently, exploring the fractional version of the FN equation is imperative, offering deeper insights into the various applications of this equation.

In the literature, numerous researchers have endeavored to find both analytical and numerical solutions to the time-fractional Fitzhugh-Nagumo (FN) equation. Cevikel AC, Bekir A, Abu Arqub O, and Abukhaled M have discussed the exact solution of the FN equation utilizing conformable fractional derivatives(7), while Ramani, P., Khan, A.M., and Suthar, D.L. employed the Reduced Differential Transform method(8). Sinan Deniz utilized the optimal perturbation iteration method to derive solutions for the FN equation utilizing AB time-fractional derivatives(9). Additionally, INan B., Ali K.K., Saha A., and Ak T applied the exponential finite difference method(10), and Berat Karaagac discussed the finite element method(11). Alam, M., Haq S., Ali I., and Ebadi M.J. employed radial basis functions(12). Bhausaheb Sontakke and Rajashri Pandit utilized the Adomian Decomposition method to obtain numerical solutions for the time-fractional FN equation(13). Zhi-Yong Fan, Khalid K. Ali, M. Maneea, Mustafa Inc, and Shao-Wen Ya compared solutions of the FN equation obtained through three different methods(14).

The homotopy perturbation technique (HPM)(15) stands out as a potent numerical approach for tackling problems characterized by small parameters. It facilitates the creation of approximate solutions through series expansion. The solution is derived as a series expansion in terms of the homotopy parameter, gradually converging to the solution of the nonlinear problem. Thus, we apply the homotopy perturbation technique (HPM) to address the Fitzhugh-Nagumo equation.

We consider the following system of time-fractional Fitzhugh–Nagumo equations,

$$\frac{\partial^\alpha z}{\partial t^\alpha} = D\frac{\partial^2 z}{\partial x^2} + z(z-1)(a-z) - w$$

$$\frac{\partial^\alpha w}{\partial t^\alpha} = bz - \alpha w$$

In this context, the variable z is directly linked to the membrane potential, while w represents a range of variables tied to elements contributing to the membrane current, including sodium, potassium, and other ions. The diffusion constant D corresponds to the axial current within the axon. The parameters $0 < a < 1$, b, and $\varepsilon$ all hold positive values. From an analytical perspective, it becomes more straightforward to comprehend the situation if we adopt the perspective that both b and $\varepsilon$ are relatively small. This assumption leads to expressions such as $b = \varepsilon L$, $\alpha = \varepsilon M$, and the condition $0 < \varepsilon \ll 1$, ultimately causing the preceding equations to transform into the following form:

$$\frac{\partial^\alpha z}{\partial t^\alpha} = D\frac{\partial^2 z}{\partial x^2} + z(z-1)(a-z) - w$$

$$\frac{\partial^\alpha w}{\partial t^\alpha} = \varepsilon(Lz - Mw)$$

In the limit as $\varepsilon \to 0$, we observe that w tends towards a constant value, and this constant is found to be zero. Consequently, in this scenario, the Fitzhugh–Nagumo system simplifies to the nonlinear reaction–diffusion equation.

$$\frac{\partial^\alpha z}{\partial t^\alpha} = D\frac{\partial^2 z}{\partial x^2} + z(z-1)(a-z) \tag{1}$$

where $0 < \alpha \leq 1$. Both a, D are unconstrained, where $0 \leq a \leq 1$ and $D > 0$. We aim to solve proposed Equation (1) by using homotopy perturbation method.

The structure of this paper is outlined as follows: Section 2 presents fundamental definitions of fractional derivatives and integrals, along with a discussion on the fractional homotopy perturbation method and an exploration of its convergence. Section 3 details numerical experiments carried out to evaluate the efficiency of the proposed method. In Section 4, the conclusion is presented. The paper concludes with a list of references in the final section.

# 2 Methodology

## 2.1 Fractional order Derivatives and Integrals

Definition 1: The Riemann-Liouville fractional integral of a function f(x) of order $\alpha > 0$ is denoted by $I^{\alpha} f(x)$ and is defined as[1]:

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} f(t) dt$$

Definition 2: The Caputo derivative of a function f(t) of order $\alpha > 0$ is denoted by $\frac{\partial^{\alpha}}{\partial t^{\alpha}} f(t)$ and is defined as[1]:

$$\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-\tau)^{n-\alpha-1} \frac{d^{n}}{d\tau^{n}} f(\tau) d\tau$$

where $\Gamma$ is the gamma function, n is the smallest integer greater than $\alpha$, a is the lower limit of integration, and $\frac{d^n}{d\tau^n} f(\tau)$ denotes the n$^{\text{th}}$ derivative of $f$ with respect to $\tau$.

**Remark**:

$$(i)\; I^{\alpha}(t^{s}) = \frac{\Gamma(s+1)}{\Gamma(\alpha+s+1)} t^{\alpha+1}, \;\; for\, s > -1,\; \alpha \geq 0$$

$$(ii)\; I^{\alpha} \frac{\partial^{\alpha}}{\partial x^{\alpha}} f(x) = f(x) - \sum_{k=0}^{r-1} f^{k}\left(0^{+}\right) \frac{x^{k}}{k!}\; x > 0,\; \alpha \geq 0$$

$$(iii)\; I^{\alpha} I^{\beta}(f(t)) = I^{\alpha+\beta}(f(t))\, for\; \alpha, \beta \geq 0$$

## 2.2 Fractional Homotopy Perturbation Method

To illustrate the core concept of this approach, we consider a general nonlinear partial differential equation of the form:

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}} z(x,t) + N z(x,t) = 0 \quad m-1 < \alpha < m \tag{2}$$

where $m \in \mathbb{N}$ the differential operator $\frac{\partial^{\alpha}}{\partial x^{\alpha}}$ represents the $\alpha^{\text{th}}$ order fractional derivative, R is a linear operator, and N is a nonlinear operator. In this method, we establish a homotopy $H(u,p) : R\, X\, [0,1] \longrightarrow R$ for the fractional partial differential Equation (2) as follows:

$$H(u,p) = (1-p)\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}} u - \frac{\partial^{\alpha}}{\partial x^{\alpha}} z_{0}\right] + p\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}} u + N u\right] = 0 \tag{3}$$

$$H(u,p) = \frac{\partial^{\alpha}}{\partial x^{\alpha}} u - \frac{\partial^{\alpha}}{\partial x^{\alpha}} z_{0} + p\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}} z_{0} + N u\right] = 0$$

Where $z_0 = z(x,0)$ serves as the initial approximation for Equation (2). Assuming that the solution of Equation (3) can be expressed as a power series in p:

$$u = \sum_{n=0}^{\infty} p^{n} u_{n} = u_{0} + p u_{1} + p^{2} u_{2} + p^{3} u_{3} + \ldots \tag{4}$$

By substituting Equation (4) into Equation (3), we derive:

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\sum_{n=0}^{\infty} p^{n} u_{n}\right) - \frac{\partial^{\alpha}}{\partial x^{\alpha}} z_{0} + p\left[\frac{\partial^{\alpha}}{\partial x^{\alpha}} z_{0} + N\left(\sum_{n=0}^{\infty} p^{n} u_{n}\right)\right] = 0 \tag{5}$$

Equating the coefficients of like powers of p, we obtain the following system of equations:

$$p^0: \frac{\partial^\alpha}{\partial x^\alpha} u_0 = \frac{\partial^\alpha}{\partial x^\alpha} z_0$$

$$p^1: \frac{\partial^\alpha}{\partial x^\alpha} u_1 + \frac{\partial^\alpha}{\partial x^\alpha} z_0 + Nu_o = 0$$

$$p^2: \frac{\partial^\alpha}{\partial x^\alpha} u_2 + Nu_1 = 0$$

$$p^3: \frac{\partial^\alpha}{\partial x^\alpha} u_3 + N\, u_2 = 0$$

$$p^4: \tfrac{\partial^\alpha}{\partial x^\alpha} u_4 + Nu_3 = 0$$

$$\vdots$$

$$\vdots$$

and so forth. Solving this system of equations yields expressions for $u_0,\ u_{1,}\ u_2,\ \ldots$ as follows:

$$\left.\begin{aligned} u_0 &= z_0 \\ u_{n+1} &= -I^{\propto}(Nu_n) \end{aligned}\right\} \tag{6}$$

Consequently, an approximate solution for Equation (2) can be obtained by setting p = 1 in Equation (4), resulting in:

$$z = \lim_{p \longrightarrow 1} u = u_0 + u_1 + u_2 + \ldots$$

### 2.3 Convergence

Equating the coefficients of like powers of p of Equation (5), we can obtain equations of the following form:

$$p^0: \frac{\partial^\alpha}{\partial x^\alpha} u_0 - \frac{\partial^\alpha}{\partial x^\alpha} z_0 = 0$$

$$p^1: \frac{\partial^\alpha}{\partial x^\alpha} u_1 + \frac{\partial^\alpha}{\partial x^\alpha} z_0 + H_0 = 0$$

$$p^2: \frac{\partial^\alpha}{\partial x^\alpha} u_2 + H_1 = 0$$

$$p^3: \frac{\partial^\alpha}{\partial x^\alpha} u_3 + H_2 = 0$$

$$p^4: \tfrac{\partial^\alpha}{\partial x^\alpha} u_4 + H_3 = 0$$

$$\vdots$$

$$\vdots$$

Where $H_n(u_0, u_1, \ldots, u_n)$ are He's polynomials, which can be estimated as:

$$H_n(u_0, u_1, \ldots, u_n) = \left(\frac{1}{n!}\frac{\partial^n}{\partial p^n} N\left(\sum_{i=0}^{n} p^i u_i\right)\right)_{p=0} \tag{7}$$

So, we obtain,

$$\left.\begin{aligned} u_0 &= z_0 \\ u_{n+1} &= -I^{\propto}(H_n) \end{aligned}\right\} \tag{8}$$

**Theorem:** Let B be a Banach space. Then we have

(i) If there exists $0 \leq \lambda < 1$, such that $\| u_n \| \leq \lambda \ \| u_{n-1} \| \ \forall \, n \in N$, *then* $\sum_{i=0}^{\infty} u_i$ obtained by Equation (8), convergence to some s $\in$ B.

(ii) $s = \sum_{n=1}^{\infty} u_n \; satisfies \; s = -I^{\propto} N(s + u_0) - z_0$.

(iii) $Equation \; s = -I^{\propto} \, N(s + u_0) - z_0$is equivalent to Equation (2).

**Proof:** Consider the sequence

$$s_0 = 0, \quad s_n = u_1 + u_2 + \ldots + u_n,$$

which is determined by the iterative scheme,

$$s_{n+1} = -I^{\propto} N_n(s_n + u_0) - z_0 \tag{9}$$

Where

$$N_n\left(\sum_{i=0}^{n} u_i\right) = \sum_{i=0}^{n} H_i, \quad n = 0, 1, 2, \ldots$$

This sequence is equivalent to the solution of Equation (8), which can be proved by mathematical induction. Now, to prove (i) we have,

$$\| s_{n+1} - s_n \| = \| u_{n+1} \| \ \lambda \| u_n \| \leq \lambda^2 \| u_{n-1} \| \leq \ldots \leq \lambda^{n+1} \ \| u_0 \|$$

Then, for any $m, n \in N, \ n \geq m$, we have,

$$\| s_n - s_m \| = \| (s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \ldots + (s_{m+1} - s_m) \|$$

$$\leq \| s_n - s_{n-1} \| + \| s_{n-1} - s_{n-2} \| + \ldots + \| s_{m+1} - s_m \|$$

$$\leq \lambda^n \| u_0 \| + \lambda^{n-1} \| u_0 \| + \ldots + \lambda^{m+1} \| u_0 \|$$

$$\leq (\lambda^n + \lambda^{n-1} + \ldots + \lambda^{m+1}) \ \| u_0 \|$$

$$\leq (\lambda^{m+1} + \cdots + \lambda^n + \ldots) \ \| u_0 \|$$

$$\leq \lambda^{m+1} (1 + \lambda + \ldots + \lambda^n + \ldots) \| u_0 \|$$

$$\leq \frac{\lambda^{m+1}}{1-\lambda} \parallel u_0 \parallel$$

Therefore, $\lim_{m,n\to\infty} \|s_n - s_m\| = 0$, which gives sequence $\{s_n\}$ is a Cauchy sequence in Banach space **B**. It should be convergent, let s $\in$ **B** such that,

$$\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} u_n = s.$$

To prove (ii), from Equation (9), we have

$$s = \lim_{n\to\infty} s_{n+1} = -I^{\alpha} \lim_{n\to\infty} N_n(s_n + u_0) - z_0$$

$$= -I^{\propto} \lim_{n\longrightarrow\infty} N_n \left(\sum_{i=0}^{n} u_i\right) - z_0$$

$$= -I^{\propto} \lim_{n\longrightarrow\infty} \left(\sum_{i=0}^{n} H_i\right) - z_0$$

$$= -I^{\propto} \sum_{i=0}^{\infty} H_i - z_0$$

Also, we have

$\sum_{i=0}^{\infty} H_i = N\left(\sum_{i=0}^{\infty} u_i\right)$.

Therefore

$$s = -I^{\propto} N\left(\sum_{i=0}^{\infty} u_i\right) - z_0$$

$$= -I^{\propto} N(s + u_0) - z_0$$

To prove (iii), we apply the operator $L^{\propto}$ to the above equation, we obtain,

$\frac{\partial^{\alpha}}{\partial x^{\alpha}}(s + z_0) = -N(s + u_0)$.

But $z_0 = u_0$. Therefore,

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}(s + z_0) + N(s + u_0) = 0$$

By considering $z = s + z_0 = \sum_{n=0}^{\infty} u_n$ we have

$$\frac{\partial^{\alpha}}{\partial x^{\alpha}}(z) + N(z) = 0$$

Hence, the solution in (ii) is the same as solution of $\frac{\partial^{\alpha}}{\partial x^{\alpha}}(z) + N(z) = 0$.

## 3 Results and Discussion

In this section, we discuss some numerical examples for the numerical solution of time-fractional Fitzhugh–Nagumo equation using homotopy perturbation method. The solution of numerical examples is represented graphically.

**Example:1** Consider the following reduced time-fractional Fitzhugh–Nagumo equation

$$\frac{\partial^\alpha z}{\partial t^\alpha} = D\frac{\partial^2 z}{\partial x^2} + z(z-1)(a-z) \tag{10}$$

with initial condition $z(x,0) = \frac{1}{1+e^{\frac{x}{2\sqrt{D}}}}$. Exact solution for the given problem at $\alpha = 1$ is

$$z(x,t) = \frac{1}{1+e^{\left(\frac{x}{2\sqrt{D}}+\left(a-\frac{1}{2}\right)t\right)}} \tag{11}$$

To solve a problem **10** (Equation (10)) by fractional homotopy perturbation method, the homotopy for the problem **10** (Equation (10)) can be represented as

$$H(u,p) = \frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha z_0}{\partial t^\alpha} + p\left[\frac{\partial^\alpha z_0}{\partial t^\alpha} - D\frac{\partial^2 u}{\partial x^2} - u(u-1)(a-u)\right] = 0 \tag{12}$$

with initial approximation $z(x,0) = \frac{1}{1+e^{\frac{x}{\sqrt{2D}}}}$. Suppose the solution of Equation (10) can be expressed as a power series in p as follows:

$$u = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots \tag{13}$$

Inserting Equation (13) in **equation (14)**, we obtain

$$u_0 = z_0 = \frac{1}{1+e^{\frac{x}{\sqrt{2D}}}}$$

$$u_1 = -I^{\propto}\left(-D\frac{\partial^2 u_0}{\partial x^2} - au_0^2 + u_0^3 + au_0 - u_0^2\right)$$

$$u_2 = -I^{\propto}\left(-D\frac{\partial^2 u_1}{\partial x^2} - 2au_0u_1 + 3u_0^2u_1 + au_1 - 2u_0u_1\right)$$

$$u_3 = -I^{\propto}\left(-D\frac{\partial^2 u_2}{\partial x^2} + 3\,u_0u_1^2 - 2au_0u_1 + 3u_0^2 - u_1^2 + au_2 - 2\,u_0u_2\right)$$

$$u_4 = -I^{\propto}\left(-D\frac{\partial^2 u_3}{\partial x^2} - 2au_1u_2 + 6u_0u_1u_2 - 2au_0u_3 + 3u_0^2u_3 - 2\,u_1u_2 + a\,u_3 - 2\,u_0u_3\right)$$

and so on. Solving above system of equation, we obtain the values of
$u_0,\ u_1,\ \ldots$ as follows:

$$u_0 = z_0 = \frac{1}{1+e^{\frac{x}{\sqrt{2D}}}}$$

$$u_1 = -\frac{(2a+1)t^\alpha e^{\left(\frac{\sqrt{2}x}{\sqrt{D}}\right)} + (2a-3)t^\alpha e^{\left(\frac{\sqrt{2}x}{2\sqrt{D}}\right)}}{2\left(e^{\left(\frac{3\sqrt{2}x}{2\sqrt{D}}\right)}\alpha(\alpha+1) + 3e^{\left(\frac{\sqrt{2}x}{\sqrt{D}}\right)}\alpha(\alpha+1) + 3\,e^{\left(\frac{\sqrt{2}x}{2\sqrt{D}}\right)}\alpha(\alpha+1) + \alpha(\alpha+1)\right)}$$

We can also find further values viz. $u_0,\ u_1,\ \ldots..$ Therefore, an approximate solution for the given problem will be

$$z = \lim_{p\longrightarrow 1} u = u_0 + u_1 + u_2 + .\ .\ .$$

$$= \frac{1}{1+e^{\frac{x}{\sqrt{2D}}}} - \frac{(2a+1)t^{\alpha}e^{\left(\frac{\sqrt{2}x}{\sqrt{D}}\right)} + (2a-3)t^{\alpha}e^{\left(\frac{\sqrt{2}x}{2\sqrt{D}}\right)}}{2\left(e^{\left(\frac{3\sqrt{2}x}{2\sqrt{D}}\right)}\alpha(\alpha+1) + 3e^{\left(\frac{\sqrt{2}x}{\sqrt{D}}\right)}\alpha(\alpha+1) + 3\,e^{\left(\frac{\sqrt{2}x}{2\sqrt{D}}\right)}\alpha(\alpha+1) + \alpha(\alpha+1)\right)}$$

We contrast the exact solution with the approximate solution, revealing are markable closeness between the two.

We compare estimated approximated solution with exact solution at $-10 \leq$x$\leq$ 10, a = 1, D= 1, t = 1, $\alpha$ = 1 in Figure 1 and observe that approximate solution is closed to exact solution.

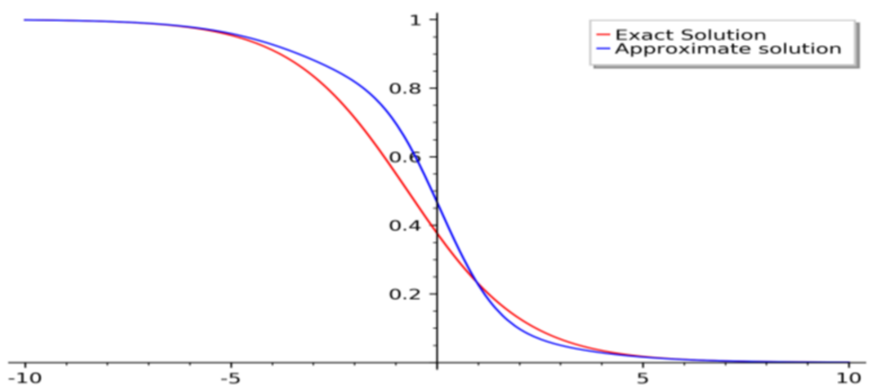


**Fig 1. Comparison of approximate solution with exact solution at $-10 \leq$ x $\leq$ 10, a = 1, D=1**

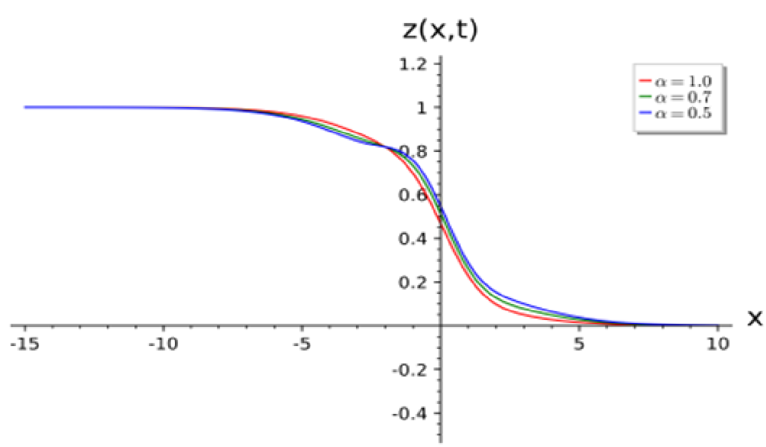


**Fig 2. Behavior of solutions at $-40 \leq$ x $\leq$ 40, a = 1, D = 0.1**

In Figure 1, we examine the behavior of solutions at $-15 \leq$ x $\leq$ 10, a = 1, D = 1, t = 1 for $\alpha$ = 0.5, 0.7, 1.0 and observed that the obtained solution is kink type travelling wave solution converges towards the solution for $\alpha$ = 1. The wave profile of the kink wave solution at t = 1, $\alpha$ = 0.9, a = 1,D = 1 is displayed in Figure 3.

**Example:2** Consider the following reduced time-fractional Fitzhugh–Nagumo equation with D = 1, a = $-1$.

$$\frac{\partial^{\alpha} z}{\partial t^{\alpha}} = \frac{\partial^2 z}{\partial x^2} - z^3 + z \tag{15}$$

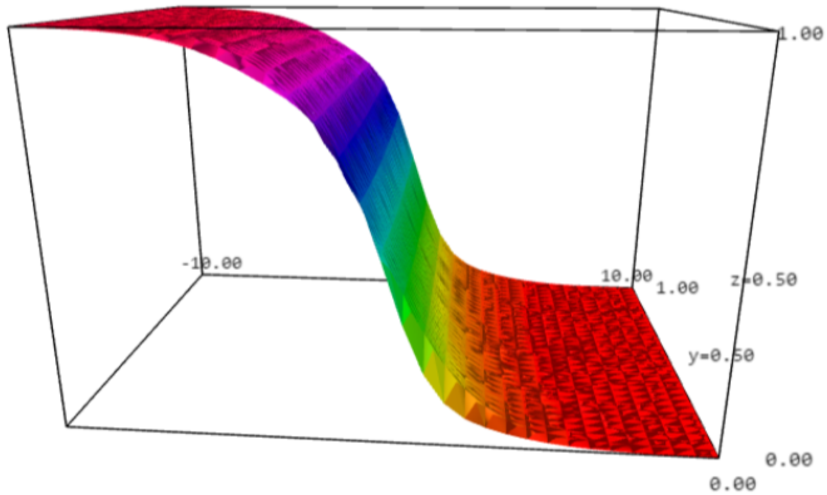


**Fig 3. Solution profile at $\alpha$= 0.9, a = 1, D= 0.5**

with initial condition $z(x,0) = \frac{1}{2}\, tanh\left(\frac{1}{4}\sqrt{2x}\right) + \frac{1}{2}$. Exact solution for the given problem at $\alpha$ = 1 is

$$z(x,t) = \frac{1}{2}\, tanh\left(\frac{1}{4}\sqrt{2x} + \frac{3}{4}t_2\right) + \frac{1}{2} \tag{16}$$

Now, we solve the above problem with similar procedure and obtain the following result

$$u_0 = z_0 = \frac{1}{2}\, tanh\left(\frac{1}{4}\sqrt{2x}\right) + \frac{1}{2}$$

$$\begin{aligned} u_1 = &\left(-1\left(0.5tanh\left(\tfrac{1}{4}\sqrt{2x}\right)\right) + 0.5\right)^2 - \left(0.5tanh\left(\tfrac{1}{4}\sqrt{2x}\right) + 0.5\right)^3 \\ &+0.125\left(tanh\left(\tfrac{1}{4}\sqrt{2x}\right)^2 - 1\right) tanh\left(\tfrac{1}{4}\sqrt{2x}\right)\left(0.5tanh\left(\tfrac{1}{4}\sqrt{2x}\right) + 0.5\right) \\ &+ \left(\left(0.5tanh\left(\tfrac{1}{4}\sqrt{2x}\right) + 0.5\right)^2\right) t^\alpha \tfrac{1}{\Gamma(\alpha+1)} \end{aligned}$$

We can also find further values viz. $u_2,\ u_{3,}\ .\ .\ .$ Therefore, an approximate solution for the given problem will be

$$z = \lim_{p \longrightarrow 1} u = u_0 + u_1 + u_2 + .\ .\ .$$

$$\begin{aligned} = &\ \tfrac{1}{2}\, tanh\left(\tfrac{1}{4}\sqrt{2x}\right) + \tfrac{1}{2} + \left(-1\left(0.5tanh\left(\tfrac{1}{4}\sqrt{2x}\right)\right) + 0.5\right)^2 - \left(0.5tanh\left(\tfrac{1}{4}\sqrt{2x}\right) + 0.5\right)^3 \\ &+ 0.125\left(tanh\left(\tfrac{1}{4}\sqrt{2x}\right)^2 - 1\right) tanh\left(\tfrac{1}{4}\sqrt{2x}\right)\left(0.5tanh\left(\tfrac{1}{4}\sqrt{2x}\right) + 0.5\right) \\ &+ \left(\left(0.5tanh\left(\tfrac{1}{4}\sqrt{2x}\right) + 0.5\right)^2\right) t^\alpha \tfrac{1}{\Gamma(\alpha+1)} \end{aligned}$$

We contrast the exact solution with the approximate solution, revealing a remarkable closeness between the two.

We compare estimated approximated solution with exact solution at $-20 \leq$ x $\leq$ 20, t = 1, $\alpha$ = 1 in Figure 4 and observe that approximate solution is closed to exact solution. In Figure 5, we examine the behavior of solutions

The wave profile of the kink wave solution at $\alpha$ = 0.9 is displayed in Figure 6.

## 4 Conclusion

The homotopy perturbation method was effectively applied to solve the time-fractional Fitzhugh-Nagumo Equation, highlighting its importance in advancing the comprehension of complex phenomena governed by this equation. We delve into the convergence of our method and find that the solutions obtained align closely with the exact solutions of the time-fractional Fitzhugh-Nagumo Equation. Additionally, our observations reveal the emergence of kink-type traveling wave solutions for the Fitzhugh-Nagumo equation.

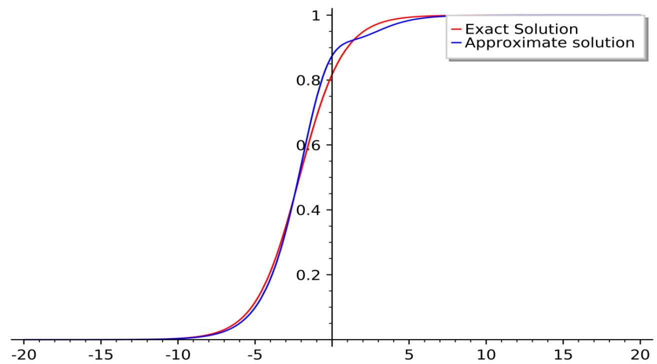


**Fig 4. Comparison of approximate solution with exact solution at $-10 \leq x \leq 10$ at $-20 \leq x \leq 20$, t = 1 for $\alpha$ = 0.5, 0.8, and 1.0 and observed that the obtained solution is kink type travelling wave solution converges towards the solution for $\alpha$ = 1**

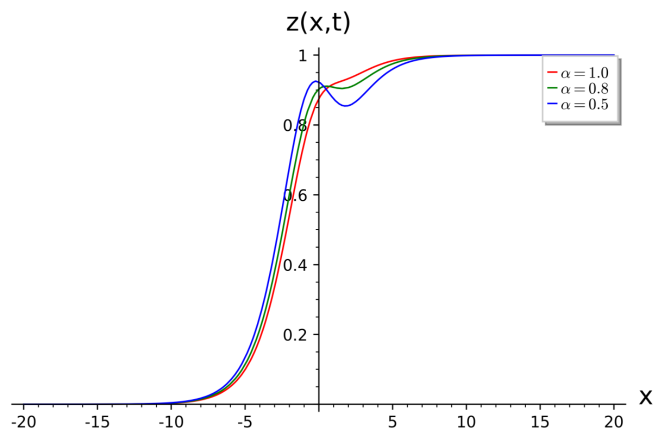


**Fig 5. Behavior of solutions at $-20 \leq x \leq 20$**

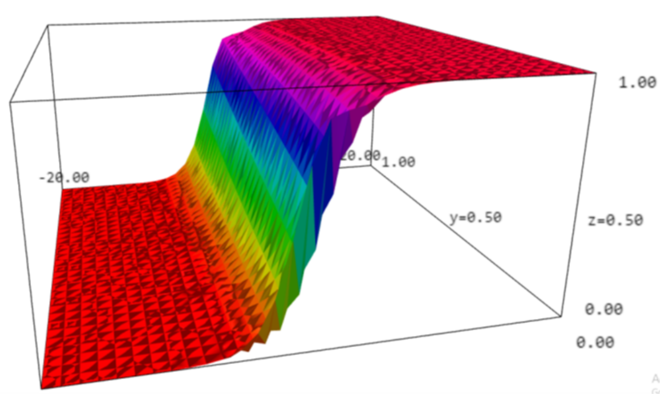


**Fig 6. Solution profile at $\alpha$ = 0.9**

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