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RESEARCH ARTICLE

Some Outcomes using Conditionally Sequential Absorbing and Pseudo Reciprocally Continuous in Multiplicative Metric Space

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Abstract

Objective: The purpose of this paper is to establish two common fixed point theorems in Multiplicative Metric Space (MMS). **Method:** By utilizing the conditions Conditionally Sequentially Absorbing (CSA) and Pseudo Reciprocally Continuous (PRC) mappings. **Findings:** Two unique common fixed point theorems are generated and supported it with relevant examples. **Novelty:** In the first theorem, we use the concept CSA and PRC mappings. In the second theorem, we use the notion of CSA, non-compatible and PRC mappings. These conditions are weaker than the existing conditions like weakly compatible mappings which generalizes the theorem of Monika Verma, P Kumar and Andnawneet Hooda.

Keywords: Multiplicative Metric Space (MMS); fixed point; Psudo Reciprocally Continuous (PRC); Conditionally Sequential Absorbing (CSA); Occasionally Weakly Compatible (OWC) mappings

1 Introduction

Fixed point theory is one of the most fascinating topics in modern mathematics and it might be the existing topics of a functional analysis. Further this topic has become a platform for numerous researchers for past many years. In the field of analysis, the theory of metric spaces has grown significantly. We know that the set of positive real numbers $\mathbb{R}^+$ is not complete in metric space. In order to overcome this problem the concept known as MMS was introduced by Bashirove in 2008. Further Ozavsar and Cevikel in 2017 investigated and developed the multiplicative contraction principle and proved some common fixed point results. Thereafter, the theory of a multiplicative metric space has been developed by many authors[1–9]. In this process Verma et al.[10] generalized the theorem proved by Cho and Yooon MMS in 2009. The purpose of this paper is to prove two common fixed point Theorems on MMS utilizing the concept like CSA and PRC mappings. Additionally, we provided two examples to validate our results. We first provide some helpful definitions and examples before presenting our findings.

2 Preliminaries

Definition 2.1 [6] Let $X \neq \varnothing$ set and $d : X \times X \to R^+$ then (X,d) is said to be MMS if it meets the requirements as below:

(i) $d(\psi,\Theta) \geq 1$ for all $\psi,\Theta \in X$ and $d(\psi,\Theta) = 1 \Leftrightarrow \psi = \Theta$

(ii) $d(\psi,\Theta) = d(\Theta,\psi)$ for all $\psi,\Theta \in X$

(iii) $d(\psi,\Theta) \leq d(\psi,\theta).d(\theta,\Theta)$ for all $\psi,\theta,\Theta \in X$ (multiplicative triangle inequality).

Definition 2.2 [6] A sequence $\{\eta_j\}$ in a MMS (X,d) is said to be

(i) multiplicative convergent sequence to η if for every multiplicative open ball $B_\delta(\eta) = \{\zeta/d(\eta,\zeta) < \delta\}, \delta > 1$, there exists a positive integer N such that $\eta_j \in B_\delta(\eta)$ for all $j \geq 1$ i.e $d(\eta_j,\eta) \to 1$ as $j \to \infty$.

(ii) multiplicative cauchy sequence if for all $\delta \geq 1 \exists N \in N$ such that $d(\eta_j,\eta_k) < \delta$ for all $j,k > N$ i.e $d(\eta_j,\eta_k) \to 1$ as $j,k \to \infty$.

Definition 2.3 [10] The pair of mapping (G,J) of a $MMS(X,d)$ is said to be

(i) Compatible if $\lim\limits_{j\to\infty} d(GJ\eta_j, JG\eta_j) = 1$ whenever a sequence $\{\eta_j\}$ in X like that $G\eta_j = J\eta_j = \zeta$ for some $\zeta \in X$.

(ii) Weakly compatible if $G\zeta = J\zeta$ for some $\zeta \in X$ such that $GJ\zeta = JG\zeta$.

(iii) OWC if the mappings commute at a coincidence point.

Example 2.1

Lt (X,d) be a MMC and $\forall \eta,\zeta \in X$ we have $d(\eta,\zeta) = e^{|\eta-\zeta|}$.Now define the self mappings G,J as $G(\eta) = 5^{\eta^3}$ and $J(\eta) = 5^{\eta^4}$ for all $\eta \in [0,1]$.

Clearly for the mappings G,J the coincidence points are $\eta = 0,1$.

At $\eta = 0$ we have $J(0) = 1 = G(0)$ and $GJ(0) = G(1) = 5^{-1}$ also $JG(0) = J(1) = 5^{-1}$

Therefore, $GJ(0) = JG(0) = 5^{-1}$. Also at $\eta = 1$ we have $G(1) = J(1) = 5^1$, $GJ(1) = \left(5^{-1}\right) = 5^{-5^{-3}}$ and $JG(1) = J\left(5^{-1}\right) = 5^{-5^{-4}}$

Therefore, $GJ(1) \neq JG(1)$. From above the maps G and J are not weakly compatible but OWC.

Definition 2.4 [5] The pair of mapping (G,J) of a $MMS(X,d)$ is said to be

(i) CSA if whenever $\{\zeta_j\}$ is a sequence satisfying $\{\zeta_j : G\zeta_j = J\zeta_j\} \neq \phi$ as $j \to \infty$ then $\exists$ another sequence $\{t_j\} \in X$ with $Gt_j = Jt_j = u$ (say) as $j \to \infty$ for some $u \in X$ such that $\lim\limits_{j\to\infty} d(Jt_j, JGt_j) = 1$ and $\lim\limits_{j\to\infty} d(Gt_j, GJt_j) = 1$.

(ii) PRC (w.r.t to CSA) whenever sequence $\{\zeta_j : G\zeta_j = J\zeta_j\} \neq \phi$ as $j \to \infty$ implies then there exists another sequence $\{\eta_j\}$ such that $G\eta_j = J\eta_j = t$ (say) then $\lim\limits_{j\to\infty} d(G\eta_j, GJ\eta_j) = 1$ and $\lim\limits_{j\to\infty} d(J\eta_j, JG\eta_j) = 1$ such that $\lim\limits_{j\to\infty} d(GJ\eta_j, Gt) = 1$ and $\lim\limits_{j\to\infty} d(JG\eta_j, Jt) = 1$.

Example 2.2

Let (X,d) be a MMS with $X = [0,10]$ and $\forall\ \eta,\zeta \in X$, $d(\eta,\zeta) = e^{|\eta-\zeta|}$.

$$G(\eta) = \begin{cases} cos\pi\eta & if\ 0 \leq \eta < \frac{1}{2} \\ \eta & if\ \frac{1}{2} \leq \eta \leq 10; \end{cases}$$

We define G,J as

$$J(\eta) = \begin{cases} cos\pi\eta + sin\pi\eta & if\ 0 \leq \eta < \frac{1}{2} \\ \eta^2 + \frac{1}{4} & if\ \frac{1}{2} \leq \eta \leq 10; \end{cases}$$

Clearly the coincidence points for the above two mappings are $= 0\ ,\ \frac{1}{2}$.

At $\eta = 0, G(0) = J(0) = 1, GJ(0) = G(1) = 1$ and $JG(0) = J(1) = 3$.

From above, it is clear that $GJ(0) \neq JG(0)$.

For a sequence $r_j = \frac{1}{j}$, for all $j \geq 1$ then

$$\lim_{j\to\infty} Gr_j = \lim_{j\to\infty} G\left(\frac{1}{j}\right) = \lim_{j\to\infty} cos\left(\frac{\pi}{j}\right) = 1 \text{ and}$$

$$\lim_{j\to\infty} Jr_j = \lim_{j\to\infty} J\left(\frac{1}{j}\right) = \lim_{j\to\infty} cos\left(\frac{\pi}{j}\right) + sin\left(\frac{\pi}{j}\right) = 1. \lim_{j\to\infty} Jr_j = \lim_{j\to\infty} J\left(\frac{1}{j}\right) = \lim_{j\to\infty} cos\left(\frac{\pi}{j}\right) + sin\left(\frac{\pi}{j}\right) = 1.$$

Therefore, $\lim\limits_{j\to\infty} Gr_j = \lim\limits_{j\to\infty} Jr_j$.

For a sequence $(t_j) = \frac{1}{2} + \frac{1}{j}$, for all $j \geq 1$. Now $\lim\limits_{j\to\infty} G(t_j) = G\left(\frac{1}{2} + \frac{1}{j}\right) = \lim\limits_{j\to\infty} \left(\frac{1}{2} + \frac{1}{j}\right) = \frac{1}{2}$ and

$$\lim_{j\to\infty} J(t_j) = J\left(\frac{1}{2} + \frac{1}{j}\right) = \lim_{j\to\infty} \left(\frac{1}{2} + \frac{1}{j}\right)^2 + \frac{1}{4} = \frac{1}{2}$$

Now $\lim\limits_{j\to\infty} GJ\left(\frac{1}{2} + \frac{1}{j}\right) = G\left(\left(\frac{1}{2} + \frac{1}{j}\right)^2 + \frac{1}{4}\right) = \frac{1}{2}$ and

$\lim_{j\to\infty} JG(t_j) = JG\left(\frac{1}{2}+\frac{1}{j}\right) = \lim_{j\to\infty} J\left(\frac{1}{2}+\frac{1}{n}\right) = \frac{1}{2}$

Therefore, $\lim_{j\to\infty} d(Gt_j, GJt_j) = 1$ and $\lim_{j\to\infty} d(Jt_j, JGt_j) = 1$. Further $\lim_{j\to\infty} d\left(GJt_j, G\left(\frac{1}{2}\right)\right) = 1$ and $\lim_{j\to\infty} d\left(JGt_j, G\left(\frac{1}{2}\right)\right) = 1$. Hence, the mappings $J,\ G$ are CSA and PSC (w.r.t CSA) but not weakly compatible.

We now go over the following example to determine how non-compatible and CSA mappings are related.

Example 2.3

Let (X,d) be a MMS with $X = [0,10]$ and $\forall \eta, \zeta \in X$ we have $d(\eta,\zeta) = e^{|\eta-\zeta|}$. We define G,J as

$G(\eta) = \begin{cases} 1+3\eta & if\ 0 \le \eta < \frac{1}{5} \\ \eta^4 & if\ \frac{1}{5} \le \eta \le 10; \end{cases}$

$J(\eta) = \begin{cases} 8\eta & if\ 0 \le \eta < \frac{1}{5} \\ \eta^2+2 & if\ \frac{1}{5} \le \eta \le 10. \end{cases}$

Let $p_j = \frac{1}{5}$, for all $j \ge 1$. Then

$$\lim_{j\to\infty} Gp_j = \lim_{j\to\infty} G\left(\frac{1}{5}\right) = \lim_{j\to\infty}\left(1+\frac{3}{5}\right) = \frac{8}{5} \tag{1}$$

and

$$\lim_{j\to\infty} Jp_j = \lim_{j\to\infty} J\left(\frac{1}{5}\right) = \lim_{j\to\infty}\frac{8}{5} = \frac{8}{5}. \tag{2}$$

From Equations (1) and (2) we get $\lim_{j\to\infty} Gp_j = \lim_{j\to\infty} Jp_j$

Therefore, $\left\{(p_j) : \lim_{j\to\infty} Gp_j = \lim_{j\to\infty} Jp_j\right\} \neq \varnothing$.

Then $\exists$ another sequence $q_j = \sqrt{2}$, for all $j \ge 1$.

$$\lim_{j\to\infty} Gq_j = \lim_{j\to\infty} G(\sqrt{2}) = \lim_{j\to\infty} 4 = 4 \tag{3}$$

and

$$\lim_{j\to\infty} Jq_j = \lim_{j\to\infty} J(\sqrt{2}) = \lim_{j\to\infty} 4 = 4. \tag{4}$$

From Equations (3) and (4), we get $\lim_{j\to\infty} Gq_j = \lim_{j\to\infty} Jq_j = 4$.

Now $\left(\lim_{j\to\infty} GJ(q_j) = GJ(\sqrt{2}) = \lim_{j\to\infty} G(4)\right) = \lim_{j\to\infty} 4^4 = 256$ and $\lim_{j\to\infty} JG(q_j) = JG(\sqrt{2}) = \lim_{j\to\infty} J(4) = 18$

Therefore, $\lim_{j\to\infty} d(GJq_j, JGq_j) \neq 1$. Hence, the mappings are non-compatible. Moreover, $\lim_{j\to\infty} d(Gq_j, GJq_j) \neq 1$ and $\lim_{j\to\infty} d(Jq_j, JGq_j) \neq 1$. Hence, the pair (G,J) is not CSA mapping.

Example 2.4

Lt (X,d) be a MMS with $X = [0,1]$ and $\forall\ \eta,\zeta \in X$ we have $d(\eta,\zeta) = e^{|\eta-\zeta|}$.

We define G,J as $G(\eta) = \begin{cases} sin(\pi\eta) & if\ 0 \le \eta < \frac{1}{2} \\ 0.9 & if\ \frac{1}{2} \le \eta \le 1 \end{cases}$ and $G(\eta) = \begin{cases} \eta\ cos(\pi\eta) & if\ 0 \le \eta < \frac{1}{2} \\ \eta & if\ \frac{1}{2} \le \eta \le 1 \end{cases}$

Let $p_j = \frac{1}{j}$, for all $j \ge 1$. Then $\lim_{j\to\infty} Gp_j = \lim_{j\to\infty} G\left(\frac{1}{j}\right) = \lim_{j\to\infty} sin\left(\frac{1}{j}\right) = 0$

and $\lim_{j\to\infty} Jp_j = \lim_{j\to\infty} J\left(\frac{1}{j}\right) = \lim_{j\to\infty} \frac{1}{j} cos\left(\frac{1}{j}\right) = 0$.

From above $\lim_{j\to\infty} Gp_j = \lim_{j\to\infty} Jp_j$

Therefore, $\left\{(p_j) : \lim_{j\to\infty} Gp_j = \lim_{j\to\infty} Jp_j\right\} \neq \varnothing$.

Then $\exists$ another sequence $q_j = 0.9$, for all $j \ge 1$.

$\lim_{j\to\infty} Gq_j = \lim_{j\to\infty} G(0.9) = \lim_{j\to\infty} 0.9 = 0.9$.

And $\lim_{j\to\infty} Jq_j = \lim_{j\to\infty} J(0.9) = \lim_{j\to\infty} 0.9 = 0.9$.

From above, we get $\lim_{j\to\infty} Gq_j = \lim_{j\to\infty} Jq_j = 0.9$.

Now $\left(\lim_{j\to\infty} GJ(q_j) = \lim_{j\to\infty} GJ(0.9) = \lim_{j\to\infty} G(0.9)\right) = \lim_{j\to\infty} 0.9 = 0.9$ and
$\lim_{j\to\infty} JG(q_j) = \lim_{j\to\infty} JG(0.9) = \lim_{j\to\infty} J(0.9) = 0.9$.

Therefore, $\lim_{j\to\infty} d(GJq_j, JGq_j) = 1$. Moreover, $d(Gq_j, GJq_j) = d(Jq_j, JGq_j) = 1$ as $j\to\infty$. Hence, the pair (G,J) is CSA as well as compatible mappings.

Verma et al.[10] established the following Theorem.

Theorem 2.1

Assume that (X,d) is a complete MMS and the mappings A, S, B and T are defined on X such that

$(B1)$ $B(X)\subseteq S(X)$ *and* $A(X)\subseteq T(X)$

(B2) $$d(A\alpha,B\beta) \le \left(max\left\{d(A\alpha,S\alpha), d(B\beta,T\beta), \sqrt{[d(A\alpha,J\beta).d(B\beta,S\alpha)]}, d(S\alpha,T\beta)\right\}\right)^p.$$
$$(max\{d(A\alpha,S\alpha), d(B\beta,T\beta)\})^q.(max\{d(A\alpha,T\beta), d(B\beta,S\alpha)\})^r$$

for all $\alpha,\beta,\in X$, where $0<p+q+2r<1$ $(p,q \text{ and } r \text{ are non}-\text{negative real numbers})$.

(B3) the subspace $A(X)$ or $B(X)$ or $S(X)$ or $T(X)$ is complete.

(B4) both the pairs $(A,\ S)$ and $(B,\ T)$ are weakly compatible.

Then the four maps A,B,S and T share a unique common fixed point in X.

We now generalize Theorem 2.1 as below.

Now we proceed to our main result.

3 Result and Discussion

3.1 Theorem

Suppose that in a complete MMS (X,d), the four self mappings A, S, B and T meeting the requirements

(C1) $B(X)\subseteq S(X)$ and $A(X)\subseteq T(X)$

(C2)
$$d(A\alpha,B\beta) \le \left(max\left\{d(A\alpha,S\alpha), d(B\beta,T\beta), \sqrt{(d[A\alpha,J\beta).d(B\beta,S\alpha)]}, d(S\alpha,T\beta)\right\}\right)^p.$$
$$(max\{d(A\alpha,S\alpha), d(B\beta,T\beta)\})^q.(max\{d(A\alpha,T\beta), d(B\beta,S\alpha)\})^r$$

for all $\alpha,\beta,\in X$, where $0<p+q+2r<1(p,q \text{ and } r \text{ are non}-\text{negative real numbers})$.

(C3) the couple $(A,\ S)$ is CSA and PRC (w.r.t. CSA) and $(B,\ T)$ is OWC.

Then there exists a unique common fixed point for the above mappings.

Proof: By $(C1)$, there is a point $u_0\in X$ such that $Au_0 = Tu_1 = v_1$.

At this $u_1\in X\exists$ a point u_2 in X $\ni Bu_1 = Su_2 = v_2$ and so on.

Likewise, we are able to define $Bu_{2j-1} = Su_{2j} = v_{2j}; Au_{2j} = Tu_{2j+1} = v_{2j+1}$ for $j = 0,1,2,\ldots$.

Now it is possible to establish that the sequence $\{v_j\}$

$$d(v_{2j+1},v_{2j+2}) \le d(Au_{2j},Bu_{2j+1})$$
$$\le \left(max\left\{d(Au_{2j},Su_{2j}), d(Bu_{2j+1},Tu_{2j+1}), \sqrt{[d(Au_{2j},Tu_{2j+1}).d(Bu_{2j+1},Su_{2j})]}, d(Su_{2j},Tv)\right\}\right)^p.$$
$$(max\{d(Au_{2j},Su_{2j}), d(Bu_{2j+1},Tu_{2j+1})\})^q.(max\{d(Au_{2j},Tu_{2j+1}), d(Bu_{2j+1},Su_{2j})\})^r$$
$$d(v_{2j+1},v_{2j+2})$$
$$\le \left(max\left\{d(v_{2j+1},v_{2j}), d(v_{2j+2},v_{2j+1}), \sqrt{[d(v_{2j+1},v_{2j+1}).d(v_{2j+2},v_{2j})]}, d(v_{2j},v_{2j+1})\right\}\right)^p.$$
$$(max\{d(v_{2j+1},v_{2j}), d(v_{2j+2},v_{2j+1})\})^q.(max\{d(v_{2j+1},v_{2j+1}), d(v_{2j+2},v_{2j})\})^r$$
$$d(v_{2j+1},v_{2j+2})$$
$$\le \left(max\left\{, d(v_{2j+1},v_{2j}),\ d(v_{2j+2},v_{2j+1}),\ \sqrt{[d(v_{2j+1},v_{2j+1}).d(v_{2j+1},v_{2j}).d(v_{2j+1},v_{2j+2}]}, d(v_{2j},v_{2j+1})\right\}\right)^p.$$
$$(max\{d(v_{2j+1},v_{2j}), d(v_{2j+2},v_{2j+1})\})^q.(max\{d(v_{2j+1},v_{2j+1}).d(v_{2j+1},v_{2j}).d(v_{2j+1},v_{2j+2})\})^r$$

In the above equation, if $d(v_{2j+2},v_{2j+1}) > d(v_{2j+1},v_{2j})$ for some positive integer j,

then $d(v_{2j+1},v_{2j+2}) \le d(v_{2j+1},v_{2j+2})^h$, where $0<h=p+q+2r<1$, a contradiction.

Therefore, we have $d(v_{2j+2},v_{2j+1}) \le d(v_{2j},v_{2j+1})^h$.

Similarly , we obtain

$d\left(v_j, v_{2j+1}\right) \leq \left(d\left(v_{j-1}, v_j\right)^h\right) \leq \left(d\left(v_{j-2}, v_{j-1}\right)^{h^2}\right) \ldots\ldots \leq d(v_0, v_1)^{h^n}$.
Let $l, j \in N$ such that $l > j$, such that
$d\left(v_l, v_j\right) \leq d\left(v_l, v_{l-1}\right) \ldots\ldots d\left(v_{j+1}, v_j\right)$
$\leq \left(d(v_1, v_0)\right)^{h^{l-1}+\ldots h^j}$
$\leq \left(d(v_1, v_0)\right)^{\frac{h^j}{1-h}} \to 1$ as $l, j \to \infty$ where $0 < h = p+q+2r < 1$.
Thus, $\{v_j\}$ is cauchy sequence.
By the completeness of $X \exists z \in X \ni v_j \to \psi$ as $j \to \infty$.
Accordingly, the sequences $Au_{2j}, Su_{2j}, Tu_{2j+1}, Bu_{2j+1} \to \psi$ as $j \to \infty$.
By (C3) the couple $(A,\ S)$ is CSA $\exists \{v_j\} \in X$ like that
$\lim_{j\to\infty} Av_j = \lim_{j\to\infty} Sv_j = \psi$ satisfying $d\left(Av_j, ASv_j\right) = 1$ and

$$d\left(Sv_j, SAv_j\right) = 1 \tag{5}$$

and by PRC (w.r.t. CSA) of the couple $(A,\ S)$ we have
$\lim_{j\to\infty} d\left(Av_j, ASv_j\right) = 1$ and $\lim_{j\to\infty} d\left(Sv_j, SAv_j\right) = 1$ such that

$$\lim_{j\to\infty} ASv_j = A\psi \text{ and } \lim_{j\to\infty} SAv_j = S\psi \tag{6}$$

Therefore, from Equations (5) and (6)

$$A\psi = S\psi = \psi. \tag{7}$$

Snice $A\psi \in A(X)$ then by (C1) $\exists\ \varphi$ like that $A\psi = T\varphi$ and hence

$$\psi = A\psi = S\psi = T\varphi \tag{8}$$

Claim $B\varphi = T\varphi$.
Putting $=\ \psi, v = \varphi$ in (C2) and using Equation (8), we get
$d(A\psi, B\varphi) \leq \left(max\left\{d(A\psi, S\psi), d(B\varphi, T\varphi), \sqrt{[d(A\psi, T\varphi).d(B\varphi, S\psi)]}, d(S\psi, T\varphi)\right\}\right)^p.$
$(max\{d(A\psi, S\psi), d(B\varphi, T\varphi)\})^q.(max\{d(A\psi, T\varphi), d(B\varphi, S\psi)\})^r.$
$d(A\psi, B\varphi) \leq \left(max\left\{d(\psi, \psi), d(B\varphi, T\varphi), \sqrt{[d(\psi, \psi).d(B\varphi, T\varphi)]}, d(\psi, \psi)\right\}\right)^p.$
$(max\{d(\psi, \psi), d(B\varphi, T\varphi)\})^q.(max\{d(\psi, \psi), d(B\varphi, T\varphi)\})^r.$
$d(T\varphi, B\varphi) \leq d(T\varphi, B\varphi)^{p+q+r}$,
Hence $T\varphi = B\varphi$.
This gives $A\psi = B\varphi = T\varphi = \psi$.
From (C3) the couple $(B,\ T)$ is OWC with $B\varphi = T\varphi$ gives $BT\varphi = TB\varphi$
Implies that $B\psi = T\psi$.
Claim $\psi = B\psi$.
Putting $u = v = \psi$ in (C2)
$d(A\psi, B\psi) \leq \left(max\left\{d(A\psi, S\psi), d(B\psi, T\psi), \sqrt{[d(A\psi, T\psi).d(B\psi, S\psi)]}, d(S\psi, T\psi)\right\}\right)^p.$
$(max\{d(A\psi, S\psi), d(B\psi, T\psi)\})^q.(max\{d(A\psi, T\psi), d(B\psi, S\psi)\})^r$
$d(\psi, B\psi) \leq \left(max\left\{d(\psi, \psi), d(T\psi, T\psi), \sqrt{[d(\psi, B\psi).d(B\psi, \psi)]}, d(\psi, B\psi)\right\}\right)^p.$
$(max\{d(\psi, \psi), d(T\psi, T\psi)\})^q.(max\{d(\psi, B\psi), d(B\psi, \psi)\})^r$
$d(\psi, B\psi) \leq d(\psi, B\psi)^{p+q+r}$
which implies $\psi = B\psi$.
Therefore,

$$B\psi = T\psi = \psi \tag{9}$$

From Equations (7) and (9) $\psi = S\psi = A\psi = T\psi = B\psi$.
This demonstrates that the common fixed point for the maps above is ψ.

For Uniqueness : Assume that ρ be another fixed point then $\rho = S\rho = A\rho = T\rho = B\rho$.
Putting $u = \psi$ and $v = \rho$ in (C2)
$d(A\psi, B\rho) \leq \left(max\left\{d(A\psi, S\psi), d(B\rho, T\rho), \sqrt{[d(A\psi, T\rho).d(B\rho, S\psi)]}, d(S\psi, T\rho)\right\}\right)^p.$
$(max\{d(A\psi, S\psi), d(B\rho, T\rho)\})^q.(max\{d(A\psi, T\rho), d(B\rho, S\psi)\})^r$
$d(\psi, \rho) \leq \left(max\left\{d(\psi, \psi), d(\rho, \rho), \sqrt{[d(\psi, \rho).d(\rho, \psi)]}, d(\psi, \rho)\right\}\right)^p.$
$(max\{d(\psi, \psi), d(\rho, \rho)\})^q.(max\{d(\psi, \rho), d(\rho, \psi)\})^r$
$d(\psi, \rho)\ d(\psi, \rho)^{p+q+r}$, a contradiction
which implies $\psi = \rho$.
This demonstrates that the unique common fixed point for the maps above is ψ.
Now we give an example to validate our result.

3.2 Example:

Let (X, d) be an MMS and $X = [2, 10]$ with $d(\beta, \eta) = e^{|\beta - \eta|}$for all $\beta, \eta \in X$.
Define self-mappings $A, B,\ S$ and T as

$$A(\eta) = B(\eta)\begin{cases} \frac{\eta+2}{2} & if\ 2\ \leq \eta < 4 \\ \frac{4\eta-1}{5} & if\ 4\ \leq \eta \leq 10 \end{cases} \quad \text{and}$$

$$S(\eta) = T(\eta)\begin{cases} \frac{\eta^2}{2} & if\ 2\ \leq \eta < 4 \\ 3\cos(\pi\eta) & if\ 4\ \leq \eta \leq 10 \end{cases}.$$

Clearly $A(X) = [2, 7.8] \cup \subseteq T(X) = [2, 8] \cup \{1\}$ and
$B(X) = [2, 7.8] \subseteq S(X) = [2, 8] \cup \{1\}$ so that (C1) is satisfied.
For a sequence $\{\eta_k\}\ as\eta_k = 4 + \frac{1}{k}$ for all $k > 0$
then $\lim_{k\to\infty} A\eta_k = A\left(4 + \frac{1}{k}\right) = \frac{4\left(4+\frac{1}{k}\right)-1}{5} = 3$
and $\lim_{k\to\infty} S\eta_k = S\left(4 + \frac{1}{k}\right) = 3cos\left(\pi\left(4 + \frac{1}{k}\right)\right) = 3.$
This gives $\lim_{k\to\infty} A\eta_k = \lim_{k\to\infty} S\eta_k = 3$ and hence as $k \to \infty$ the limit of $A\eta_k$ and $S\eta_k$ is non empty so that there is one more sequence. $\beta_k = 2 + \frac{1}{k} \forall\ k > 0$.
Then $\lim_{k\to\infty} A\beta_k = \lim_{k\to\infty} A\left(2\ + \frac{1}{k}\right) = \lim_{k\to\infty}\left[\frac{2\ + \frac{1}{k} + 2}{2}\right] = 2$ and $\lim_{k\to\infty} S\beta_k = \lim_{k\to\infty} S\left(2\ + \frac{1}{k}\right) = 2.$
This gives $\lim_{k\to\infty} A\beta_k = \lim_{k\to\infty} S\beta_k = 2.$
Further $\lim_{k\to\infty} AS\beta_k = \lim_{k\to\infty} A\left(\frac{1}{2}\left(2\ + \frac{1}{k}\right)^2\right) = \lim_{k\to\infty} A(2) = 2$
and $\lim_{k\to\infty} SA\beta_k = \lim_{k\to\infty} SA\left(2\ + \frac{1}{k}\right) = \lim_{k\to\infty} S(2) = 2.$
Thus, from the above $\lim_{k\to\infty}\ AS\beta_k = \lim_{k\to\infty} A\beta_k$ and $\lim_{k\to\infty}\ SA\beta_k = \lim_{k\to\infty} S\beta_k$.
Also, $\lim_{k\to\infty}\ AS\beta_k = A(2)$ and $\lim_{k\to\infty}\ SA\beta_k = S(2)$.
Therefore, the couple (A, S) is CSA and PRC (w.r.t CSA).
Also, $A(2) = S(2) = 2$ and $B(2) = T(2) = 2$,
seeing that 2,4 are the coincidence point of $A,\ S$
But $AS(2) = A(2) = 2$ and $SA(2) = S(2) = 2$.
Therefore, $AS(2) =\ SA(2)$.
Similarly, BT(2) $=\ TB(2)$.
Also, at $\eta = 4$ we have $A(4) = S(4) = 3$.
But $AS(4) = A(3) = \frac{5}{2}$ and $SA(4) = S(3) = \frac{9}{2}$.
Therefore, $AS(4) \neq\ SA(4)$. Similarly, $BT(4) \neq\ TB(4)$.
Showing that the couple (A, S) and (B, T)) are OWC but not WCM.
It can be observed from the self maps defined in the above example
$A(2) = S(2) = B(2) = T(2) = 2$, demonstrating that the unique common fixed point is 2.
We now demonstrate another theorem on MMS.

3.3 Theorem:

The following conditions are assumed to be met by the mappings A , S , B and T where (X,d) is a complete MMS

(D1) $B(X) \subseteq S(X)$ $and\ A(X) \subseteq T(X)$

(D2)

$$d(A\alpha,B\beta) \leq \left(max\left\{d(A\alpha,S\alpha),d(B\beta,T\beta),\sqrt{[d(A\alpha,J\beta).d(B\beta,S\alpha)]},d(S\alpha,T\beta)\right\}\right)^p. (max\{d(A\alpha,S\alpha),d(B\beta,T\beta)\})^q.(max\{d(A\alpha,T\beta),d(B\beta,S\alpha)\})^r$$

for all $\alpha,\beta,\in X$, where $0 < p+q+2r < 1$ $(p,q\ and\ r\ are\ non-negative\ real\ numbers)$.

(D3) the couples $(A,\ S)$ and $(B,\ T)$ are non-compatible PRC (w.r.t. CSA) and CSA.

Then there exists a unique common fixed point for the above mappings.

Proof: As in theorem (3.1), $\{v_j\}$ is cauchy sequence.

From (D3) the couple (A,S) is non-compatible $\exists\{u_j\} \in X$ with

$\lim_{j\to\infty} Au_j = \lim_{j\to\infty} Su_j = \psi$ for some $\psi \in X$but either $\lim_{j\to\infty} d(ASu_j,SAu_j) \neq 1$ or the limit is non-existent.

Also, the couple (A,S) is CSA then $\exists$ a sequence $\{v_j\} \in X$ like that

$\lim_{j\to\infty} Av_j = \lim_{j\to\infty} Sv_j = \psi$ (say) with $\lim_{j\to\infty} d(Av_j,ASv_j) = 1$ and $\lim_{j\to\infty} d(Sv_j,ASv_j) = 1$.

Now by PRC (w.r.t. CSA) of the pair (A,S) we have

$\lim_{j\to\infty} d(A\psi,ASv_j) = 1$ and $\lim_{j\to\infty} d(S\psi,ASv_j) = 1$

By the above limits, we get,

$$A\psi = S\psi = \psi \tag{10}$$

Since the non-compatible of the couple $(B,\ T)$ implies $\exists$ a sequence $\{u_j\} \in X$ like that

$\lim_{j\to\infty} Bu_j = \lim_{j\to\infty} Tu_j = \varphi$ for some $\varphi \in X$ but either $\lim_{j\to\infty} d(BTu_j,TBu_j) \neq 1$or the limit is non-existent.

Also the pair $(B,\ T)$ is CSA, $\exists$ a sequence $\{v_j\} \in X$ like that

$\lim_{j\to\infty} Bv_j = \lim_{j\to\infty} Tv_j = \beta$with $\lim_{j\to\infty} d(Bv_j,BTv_j) = 1$ and $\lim_{j\to\infty} d(Tv_j,TBv_j) = 1$.

Now by PRC (w.r.t. CSA) of the couple $(B,\ T)$ we have

$\lim_{j\to\infty} d(Bv_j,BTv_j) = 1$ and $\lim_{j\to\infty} d(Tv_j,TBv_j) = 1$.

By the above limits, we get

$$B\beta = T\beta = \beta. \tag{11}$$

Let $\beta \neq \psi$.

Substituting $u = \psi$ and $v = \beta$ in (D2) we get

$d(A\psi,B\beta) \leq \left(max\left\{d(A\psi,S\psi),d(B\beta,T\beta),\sqrt{[d(A\psi,T\beta).d(B\beta,S\psi)]},d(S\psi,T\beta)\right\}\right)^p.$

$(max\{d(A\psi,S\psi),d(B\beta,T\beta)\})^q.(max\{d(A\psi,T\beta),d(B\beta,S\psi)\})^r.$

$d(\psi,\beta) \leq \left(max\left\{d(\psi,\psi),d(\beta,\beta),\sqrt{[d(\psi,\beta).d(\beta,\psi)]},d(\psi,\beta)\right\}\right)^p.$

$(max\{d(\psi,\psi),d(\beta,\beta)\})^q.(max\{d(\psi,\beta),d(\beta,\psi)\})^r.$

$d(\psi,\beta) \leq d(\psi,\beta)^{p+r}$, a contradiction.

Since $d(\psi,\beta)^{1-(p+r)} \leq 1$

which gives $d(\psi,\beta) = 1$ since $d(\psi,\beta) \geq 1$ in MMS

Therefore, $d(\psi,\beta) = 1$

Hence, $\psi = \beta$.

Hence, from Equations (10) and (11) we get $A\psi = S\psi = B\psi = T\psi = \psi$.

This makes ψ the fixed point of four self mappings.

For Uniqueness : Assume that ρ be another fixed point then $\rho = S\rho = A\rho = T\rho = B\rho$.

Putting $u = \psi$ and $v = \rho$ in (C2)

$d(A\psi,B\rho) \leq \left(max\left\{d(A\psi,S\psi),d(B\rho,T\rho),\sqrt{[d(A\psi,T\rho).d(B\rho,S\psi)]},d(S\psi,T\rho)\right\}\right)^p.$

$(max\{d(A\psi,S\psi),d(B\rho,T\rho)\})^q.(max\{d(A\psi,T\rho),d(B\rho,S\psi)\})^r$

$d(\psi,\rho) \leq \left(max\left\{d(\psi,\psi),d(\rho,\rho),\sqrt{[d(\psi,\rho).d(\rho,\psi)]},d(\psi,\rho)\right\}\right)^p.$

$(max\{d(\psi,\psi), d(\rho,\rho)\})^q.(max\{d(\psi,\rho), d(\rho,\psi)\})^r$
$d(\psi,\rho)\ d(\psi,\rho)^{p+q+r}$, a contradiction
which implies $\psi = \rho$.
This demonstrates that the unique common fixed point for the maps above is ψ.
Now, we provide an example to support our result.

3.4 Example:

Suppose $X = [0,10]$is defined in an MMS and $d(\beta,\eta) = e^{|\beta-\eta|}$ for all $\beta,\eta \in X$
Define self-mappings A, B, S and Tas
$A(\eta) = B(\eta)\begin{cases} \frac{\eta+1}{2} & if\ 0 \le \eta \le 1 \\ 2 & if\ 1 < \eta \le 10 \end{cases}$ and $S(\eta) = T(\eta) = \begin{cases} \eta & if\ 0 \le \eta \le 1 \\ \log\eta & if\ 1 < \eta \le 10 \end{cases}$.
Clearly $A(X) = [0.5,1] \cup \{2\} \subseteq T(X) = [0, log10]$, $B(X = [0.5,1] \cup \{2\}) \subseteq S(X) = [0, log10]$ so that (D1) is satisfied.
Now take a sequence $\{\eta_j\}\ as\eta_j = e^2 + \frac{1}{j}$ for all $j \ge 1$.
Then $\lim_{j\to\infty} A\eta_j = \lim_{k\to\infty} A\left(e^2 + \frac{1}{j}\right) = \lim_{j\to\infty} 2 = 2$ and $\lim_{j\to\infty} S\eta_j = S\left(e^2 + \frac{1}{j}\right) = 2$.
This implies $\lim_{j\to\infty} A\eta_j = \lim_{j\to\infty} S\eta_j = 2$.
Also, we have $\lim_{j\to\infty} AS\eta_j = \lim_{j\to\infty} AS\left(e^2 + \frac{1}{j}\right) = \lim_{j\to\infty} A\left(log\left(e^2 + \frac{1}{j}\right)\right) = \lim_{k\to\infty} 2 = 2$
and $\lim_{j\to\infty} SA\eta_j = \lim_{j\to\infty} SA\left(e^2 + \frac{1}{j}\right) = \lim_{j\to\infty} S(2) = log2$.
$\lim_{j\to\infty} d(AS\eta_j, SA\eta_j) = d(2, log2) = e^{(2-log2|} \neq 1$.
Similarly, $\lim_{j\to\infty} d(BT\eta_j, TB\eta_j) = d(2, log2) = e^{(2-log2|} \neq 1$.
Hence, the couples (A,S) and (B,T) are non-compatible.
Now for a sequence $\beta_j = 1 - \frac{1}{j}$ for all $j > 0$.
Then $\lim_{j\to\infty} A\beta_j = \lim_{j\to\infty} A\left(1 - \frac{1}{j}\right) = 1$ and $\lim_{j\to\infty} S\beta_j = \lim_{j\to\infty} S\left(1 - \frac{1}{j}\right) = 1$.
This gives $\lim_{j\to\infty} A\beta_j = \lim_{j\to\infty} S\beta_j = 1$.
Now $\lim_{j\to\infty} AS\beta_j = \lim_{j\to\infty} AS\left(1 - \frac{1}{j}\right) = \lim_{j\to\infty} A\left(1 - \frac{1}{j}\right) = 1$
and $\lim_{j\to\infty} SA\beta_j = \lim_{j\to\infty} SA\left(1 - \frac{1}{j}\right) = \lim_{k\to\infty} S\left(1 - \frac{1}{j}\right) = 1$.
Therefore, $\lim_{j\to\infty} d(AS\beta_j, A\beta_j) = d(1,1) = e^{|1-1|} = 1$
and $\lim_{j\to\infty} d(SA\beta_j, S\beta_j) = d(1,1) = e^{|1-1|} = 1$.
Further $\lim_{j\to\infty} d(AS\beta_j, A(1)) = d(1,1) = e^{|1-1|} = 1$
and $\lim_{j\to\infty} d(SA\beta_j, S(1)) = d(1,1) = 1$.
Hence, the couple (A, S) is CSA and PRC (w.r.t. CSA).
Similarly, the couple (B, T) CSA and PRC (w.r.t. CSA).
Further $A(e^2) = S(e^2) = 2$ and $B(e^2) = T(e^2) = 2$.
But $AS(e^2) = A(2) = 2$ and $SA(e^2) = S(2) = log2$.
Therefore, $AS(e^2) \neq SA(e^2)$. Similarly,$BT(e^2) \neq TB(e^2)$.
But at $\eta = 1$ we have $A(1) = S(1) = 1$
so that $AS(1) = A(1) = 1$ and $SA(1) = S(1) = 1$.
Therefore, $AS(1) = SA(1)$. Similarly, $BT(1) = TB(1)$.
This implies the couples (A, S) and (B,T) are OWC but not WCM.
It can be observed from the self maps defined in the above example
$A(1) = S(1) = B(1) = T(1) = 1$, proving that the unique common fixed point is 1.

4 Conclusion

In this study, the concepts of CSA, PRC and OWC mappings are used for obtaining the generalization of existing common fixed point theorems proved in [10] which are weaker than those classes of WCM and continuous mappings. Further, these results are also substantiated with appropriate examples.

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