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RESEARCH ARTICLE

On Goethals and Seidel Array

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Abstract

Objectives: In this article, we aim to find a series of Hadamard matrices by suitable selection of the special class of matrices given in the Goethals and Seidel array and study the pattern obtained. **Methods**: In the presented work, the search technique of Hadamard matrices has been done by selecting special class of (0,1) negacyclic matrices instead of the back diagonal identity matrix given in Geothals and Seidel arrays and the possible existence of negacyclic matrices for the corresponding four matrices have been explored in each case. **Findings**: Corresponding to the special class of (0,1) negacyclic matrices, no sets of four negacyclic matrices have been obtained in the Goethal Seidel array, for even orders. For odd orders, except in the case when all four matrices are equal and the case when there is a pair of equal matrices, many outputs have been obtained for the remaining cases, the search domain being upto 11,9 and 7 for the case of two different, three different and four different matrices respectively, in the Goethal Seidel array. **Novelty**: The selection of special class of negacyclic matrices instead of the back diagonal identity matrix and finding the corresponding four negacyclic matrices in Goethals and Seidel arrays for constructing Hadamard matrices provides a new approach to the construction of Hadamard matrices.

Keywords: Hadamard matrix; Circulant matrix; Williamson matrices; Orthogonal array; Goethals and Seidel array

1 Introduction

In previous researches focusing on constructing Hadamard matrices using circulant matrix blocks, it was discovered that certain skew-Hadamard matrices couldn't be formed, even when dealing with relatively small orders, like 36 [(1)]. In order to address this, prob, Goethals and Seidel created the Goethal Seidel array (GS array) [(1)]. The GS array is defined as follows:

$$GS = \begin{bmatrix} A & BR & CR & DR \\ -BR & A & -D^TR & C^TR \\ CR & D^TR & A & -B^TR \\ -DR & -C^TR & -B^TR & A \end{bmatrix}$$

where R is the back diagonal identity matrix and, A, B, C, and D are ± 1 circulant matrices of order n such that $AA^T + BB^T + CC^T + DD^T = 4nI_n$. Taking A, B, C, D also to be symmetric, a Williamson array results, given by the following matrix:

$$W = \begin{bmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{bmatrix}$$

So, GS array is similar to Williamson array but with weaker conditions on A, B, C and D[2]. Using Goethals-Seidel difference families with a repeated block, a wide collection of symmetric and skew-symmetric Hadamard matrices have been obtained by Abuzina et.al[3]. Generating skew Hadamard matrices through Goethals and Seidel method, using best matrices is a notable approach in the construction of Hadamard matrices. Best matrices exist for all orders of the type $r^2 + r + 1$, which have been verified upto r=6[4].

By a method known as The Turyn based quantum computing Method, Suksmono and Minato applied it to find a set of 4-sequences {X,Y,Z,W} having particular properties, which are then used to construct a H-matrix based on Goethals-Seidel method[5]. Álvarez et.al have used Goethals-Seidel loops so that every Hadamard matrix of Goethals-Seidel type pseudococyclically developed over them[6].

2 Methodology

In the GS array corresponding to each order n, we have considered the following (n − 1), matrices R_i for (i = 1,2, … , (n − 1)) where R_i is negacyclic with first row in which 1 is at $(i + 1)^{th}$ place, and rest all entries zero. And corresponding to each such R = R_i for a particular n, we searched for negacyclic matrices A, B, C and D such that the GS array becomes a Hadamard matrix.

3 Results and discussion

Corresponding to each n and the R_i's therein, we obtained several negacyclcic matrices *A, B, C, D* (except in the case when A=B=C=D and the case when A=B & C=D) such that GS becomes a Hadamard matrix. We have mentioned just one output for each case in the tables below:

Tables:

1. For A=B=C=D, No outputs (Checked up to n=13).
2. For A=B and C=D, No outputs (Checked up to n=11).
3. For A, B distinct and B=C=D (Checked up to n=11).

Table 1.

N	First row of R	First row of A, B
3	(0,1,0)	A=(1,−1,1), B=(1,−1,−1)
	(0,0,1)	A=(1,−1,1), B=(1,−1,−1)
4,5,6,7,8,9,10,11	(0,1,0)	No output

4. For A,B,C distinct and C=D, (Checked up to n=9).

Table 2.

N	First row of R	First row of A,B,C
3	(0,1,0)	A=(1,−1,1), B=(1,−1,−1), C=(1,1,−1)
	(0,0,1)	A=(1,−1,−1), B=(1,−1,1), C=(1,1,−1)
5	(0,1,0,0,0)	A=(1,−1,1,1,−1), B=(1,1,1,−1,−1), C=(1,1,−1,1,−1)
	(0,0,1,0,0)	A=(1,1,−1,1,1), B=(1,1,1,−1,−1), C=(1,1,−1,1,−1)
	(0,0,0,1,0)	A=(1,−1,−1,−1,−1), B=(1,−1,−1,1,1), C=(1,1,−1,1,−1)),
	(0,0,0,0,1)	A=(1,1,−1,−1,−1), B=(1,−1,−1,1,1), C=(1,1,−1,1,−1)
7	(0,1,0,0,0,0,0)	A=(1,−1,1,1,1,−1,−1), B=(1,1,1,1,−1,−1,−1), C=(1,−1,1,1,−1,−1,1)

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Table 2 continued

	(0,0,1,0,0,0,0)	A=(1,−1,−1,−1,1,1,−1), B=(1,−1,1,1,−1,−1,1), C=(1,1,1,−1,1,−1,−1))]
	(0,0,0,1,0,0,0)	A=(1,−1,−1,−1,−1,−1,1), B=(1,−1,−1,1,−1,1,1), C=(1,−1,−1,−1,1,1,1))]
	(0,0,0,0,1,0,0)	A=(1,−1,−1,−1,−1,−1,−1), B=(1,−1,−1,−1,1,1,1), C=(1,−1,1,1,−1,−1,1)
	(0,0,0,0,0,1,0)	A=(1,1,−1,−1,−1,−1,−1), B=(1,−1,−1,−1,1,1,1), C=(1,−1,1,1,−1,−1,1)
	(0,0,0,0,0,0,1)	A=(1,1,1,−1,−1,−1,−1), B=(1,−1,−1,−1,1,1,1), C=(1,−1,1,1,−1,−1,1)
9	(0,1,0,0,0,0,0,0,0)	A=(1,−1,1,1,1,−1,1,−1,−1), B=(1,1,1,1,−1,1,−1,−1,−1), C=(1,1,−1,−1,1,−1,1,1,−1)
	(0,0,1,0,0,0,0,0,0)	A=(1, 1, -1, 1, 1, 1, -1, 1, -1), B=(1, 1, 1, 1, -1, 1, -1, -1, -1), C=(1, 1, -1, -1, 1, -1, 1, 1, -1)
	(0,0,0,1,0,0,0,0,0)	A=(1, -1, 1, -1, 1, -1, 1, 1, -1), B=(1, 1, -1, 1, 1, -1, -1, 1, -1), C=(1, -1, -1, -1, -1, 1, 1, 1, 1)
	(0,0,0,0,1,0,0,0,0)	A=(1, -1, -1, -1, -1, -1, -1, -1, 1), B=(1, -1, -1, -1, 1, -1, 1, 1, 1), C=(1, 1, -1, -1, 1, -1, 1, 1, -1)
	(0,0,0,0,0,1,0,0,0)	A=(1, -1, -1, 1, -1, -1, -1, 1, -1), B=(1, -1, 1, -1, -1, 1, 1, -1, 1), C=(1, -1, -1, -1, -1, 1, 1, 1, 1)
	(0,0,0,0,0,0,1,0,0)	A=(1, -1, 1, -1, -1, -1, -1, -1, -1), B=(1, -1, -1, -1, 1, -1, 1, 1, 1), C= (1, 1, -1, -1, 1, -1, 1, 1, -1)
	(0,0,0,0,0,0,0,1,0)	A=(1, 1, -1, 1, -1, -1, -1, -1, -1), B=(1, -1, -1, -1, 1, -1, 1, 1, 1), C= (1, 1, -1, -1, 1, -1, 1, 1, -1)
	(0,0,0,0,0,0,0,0,1)	A=(1, 1, 1, -1, 1, -1, -1, -1, -1), B= (1, -1, -1, -1, 1, -1, 1, 1, 1), C= (1, 1, -1, -1, 1, -1, 1, 1, -1)

5. For A,B,C,D distinct (Checked up to n=7).

Table 3.

N	First row of R	First row of A,B,C,D
3	(0,1,0)	A=(1,−1,1), B=(1,−1,−1), C=(1,1,−1), D=(1,1,1)
	(0,0,1)	A=(1,−1,1), B=(1,−1,−1), C=(1,1,−1), D=(1,1,1)
5	(0,1,0,0,0)	A=(1,1,1,−1,1), B=(1,1,1,−1,−1), C=(1,1,−1,1,−1), D=(1,−1,−1,1,1)
	(0,0,1,0,0)	A=(1,−1,−1,−1,1), B=(1,1,1,−1,−1), C=(1,1,−1,1,−1), D=(1,−1,−1,1,1)
	(0,0,0,1,0)	A=(1,−1,1,1,1), B=(1,1,1,−1,−1), C=(1,1,−1,1,−1), D=(1,−1,−1,1,1)
	(0,0,0,0,1)	A=(1,−1,1,−1,−1), B=(1,1,1,−1,−1), C=(1,1,−1,1,−1), D=(1,−1,−1,1,1)
7	(0,1,0,0,0,0,0)	A=(1,−1,1,1,1,−1,−1), B=(1,1,1,−1,1,−1,−1), C=(1,−1,−1,1,−1,1,1), D=(1,−1,−1,−1,1,1,1)
	(0,0,1,0,0,0,0)	A=(1,−1,−1,−1,1,1,−1), B=(1,1,1,1,−1,−1,−1), C=(1,1,−1,1,−1,1,−1), D=(1,−1,−1,1,−1,1,1)
	(0,0,0,1,0,0,0)	A=(1,−1,−1,−1,−1,−1,1), B=(1,1,1,−1,1,−1,−1), C=(1,1,−1,−1,1,1,−1), D=(1,−1,1,1,−1,−1,1)
	(0,0,0,0,1,0,0)	A=(1,−1,−1,−1,−1,−1,−1), B=(1,1,−1,1,−1,1,−1), C=(1,1,−1,−1,1,1,−1), D=(1,−1,−1,1,−1,1,1)
	(0,0,0,0,0,1,0)	A=(1,1,−1,−1,−1,−1,−1), B=(1,1,−1,1,−1,1,−1), C=(1,1,−1,−1,1,1,−1), D=(1,−1,−1,1,−1,1,1)

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Table 3 continued

	(0,0,0,0,0,0,0,0,1)	A=(1,1,1,−1,−1,−1,−1), B=(1,1,−1,1,−1,1,−1), C=(1,1,−1,−1,1,1,−1), D=(1,−1,−1,1,−1,1,1)

Though GS arrays were created for the construction of skew-Hadamard matrices, none of the Hadamard matrices we obtained were skew. There were no outputs for n = 2,4,6,8,10. So, it appears that for n even there does not exist any such negacyclic matrices A, B, C, D. For the case A, B, C distinct and $C = D$, outputs were obtained for n = 3,5,7,9; so, it seems that for each n odd and each corresponding R's we'll have negacyclic matrices A, B, C. For the case A, B, C, D distinct, outputs were obtained for n = 3,5,7; so here also it seems that for each n odd and each corresponding R's we'll have such negacyclic matrices A, B, C, D. The data generated may be useful for researchers trying to find patterns in the construction of Hadamard matrices.

4 Conclusion

The selection of special class of negacyclic matrices instead of the back diagonal identity matrix in the G S arrays is a new approach to search for Hadamard matrices. Hadamard matrices have been obtained in the cases mentioned above with the orders mentioned therein, using four corresponding negacyclic matrices instead of the four circulant matrices in the GS array. Using computers with higher resources and computing ability we can go higher up the orders and may be able to generalize and draw some definite conclusions from them.

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