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The Euler Characteristic of Parabolic Sheaves

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Abstract

Objectives: The primary aim of this study is to explicitly determine the Euler characteristic of the parabolic sheaves with rank 2 on a smooth projective algebraic surface defined over complex numbers $\mathbb{C}$ with the smooth irreducible parabolic divisor D. **Methods**: The computation of the parabolic Hilbert polynomial involves the use of $\mathbb{R}$-filtered sheaves on a smooth projective surface X, with weights corresponding to the points where the filtration jumps. The Riemann-Roch theorem and Chern class computation have also been used. **Findings**: The study provides explicit computations of the parabolic Hilbert polynomial as well as the parabolic Chern classes for parabolic rank 2 bundles. **Novelty**: This work contributes to the understanding of parabolic sheaves on smooth projective surfaces, bridging the gap between different constructions of stable bundles. The explicit computation of the parabolic Hilbert polynomial for rank 2 bundles adds valuable insights to the study of moduli spaces of parabolic bundles.

Keywords: Euler characteristic; Hilbert polynomial; Chern class; Parabolic sheaves; Smooth projective algebraic surface

1 Introduction

Understanding moduli spaces is crucial because we often deal with families of spaces that can transform into another space, known as the moduli space of the family. Each point in the moduli space corresponds to a space in its own right. The concept of a ”moduli problem” is central to modern algebraic geometry and is closely linked to physics. This connection necessitates a grasp of the specific parameters associated with these spaces. To navigate the complexities of moduli spaces, a thorough understanding of these parameters is essential. One particularly crucial parameter is the Chern classes, which hold significant importance in the field of algebraic geometry.

Chern classes play a pivotal role as they provide a way to quantify and characterize the geometric features of vector bundles over these spaces. These classes offer essential tools for understanding the topology and geometry of the underlying spaces, contributing to the broader study of algebraic geometry. In the realm of physics, Chern classes find applications in describing characteristic classes of fiber bundles, making them indispensable for bridging the gap between abstract mathematics and the physical world.

This work is dedicated to unraveling the nuances of Chern classes, focusing on specific cases within moduli spaces. The goal is to make these intricate concepts more accessible and provide valuable insights into the realm of algebraic geometry. A detailed description of the geometry of the moduli spaces has been discussed in [1].

The parabolic sheaves over curves were introduced by Mehta and Seshadri. This has been extended to the higher dimension by Maruyama-Yokogawa [2], Biswas, and others. Also, Li, Steer-Wren, Mochizuki, and others extended vector bundles to higher dimensions.

Assume that over $\mathbb{C}$, X is a smooth projective surface. Assume that D is an irreducible smooth divisor on X. A parabolic sheaf on X is an $\mathbb{R}$-filtered sheaf $E_\alpha := E(\alpha)$ for $\alpha \in \mathbb{R}$ (cf. Definition [2.1.1]). The weights are the real numbers. It is generally assumed that the jump happens at rational points.

The study of moduli for the parabolic bundles requires the computation of the Hilbert polynomial, which is an Euler characteristic for the parabolic sheaves twisted by a large power of an ample line bundle. On X, consider an ample line bundle H. We write $M_H^{par}(c_*)$ for the Gieseker moduli of parabolic bundles with respect to the polarization H. In [3], $M_H^{par}(c_*)$ the Donaldson-Uhlenbeck compactification of the moduli space of parabolic μ-stable bundles was constructed. Also, the construction of the parabolic analogue of a Gieseker-Uhlenbeck morphism for the moduli space of parabolic bundles on X with the parabolic structure defined on D has been discussed by R. Parthasarathi. A natural compactification over Riemann Surface by smooth divisors has been discussed in [4].

To understand the moduli space of parabolic stable sheaves on algebraic surfaces in a differential geometric point of view, we obtained the Donaldson-Uhlenbeck compactification for the parabolic stable sheaves using Γ-Categorical methods.

The moduli space of parabolic sheaves on algebraic surfaces using Maruyama-Yokogawa construction naturally contains the parabolic μ -stable bundles as a large open set. It is yet to be studied the relation between the isomorphism classes of χ-stable bundles coming from the MY-construction and the isomorphism classes of χ-stable bundles coming from the Γ-category.

In order to understand the relation, it is necessary to explicitly compute the parabolic Hilbert polynomial of the parabolic torsion-free sheaves coming from Maruyama-Yokogawa. The Verlinde formula, an expression for the Euler characteristic of line bundles on the moduli spaces of stable bundles on a curve, has been discussed in [5].

The aim of this study is to compute the parabolic Hilbert polynomial for the parabolic rank 2 bundle explicitly.

2 Methodology

We discuss some basic concepts related to the idea of a parabolic bundle. For the purpose of this paper, X will denote a smooth projective surface over the field of complex numbers $\mathbb{C}$. We denote its structure sheaf by O_X and its canonical sheaf by K_X. Let $D \subset X$ be a smooth irreducible parabolic divisor, and $O_X(1)$ be a very ample invertible sheaf on X. For a coherent sheaf E on X, the image of $E \otimes_{O_X} O_X(-D)$ in E will be denoted by $E(-D)$.

2.1 The class of bundles with parabolic structures

Definition 2.1.1 [3] Let E be a torsion-free O_X-coherent sheaf on X. Consider a divisor D on X that is effective. For a coherent sheaf E on X, the image of $E \otimes_{O_X} O_X(-D)$ in E will be denoted by $E(-D)$. A quasi-parabolic structure on E over D is a filtration by O_X-coherent subsheaves

$$E = F_1(E) \supset F_2(E) \supset \ . \ . \ . \ \supset F_l(E) \supset F_{l+1}(E) = E(-D)$$

The integer l denotes the length of the filtration. A parabolic structure is a quasi-parabolic structure, as above, together with a system of weights $\{\alpha_1,\ \alpha_2,\ .\ .\ .\ .\ ,\ \alpha_l\}$ such that,

$$0 \le \alpha_1 < \ \alpha_2 < \ . \ . \ . \ < \ \alpha_l < 1,$$

where the weight α_i corresponds to the subsheaf $F_i(E)$.

Additionally, we will only look at parabolic structures with rational weights. The parabolic sheaf will be indicated by (E_*, F_*, α_*). When there is no scope for confusion, it will be denoted by E_*.

For a parabolic sheaf (E_*, F_*, α_*) define the following filtration $\{E_t\},\ t \in \mathbb{R}$ of coherent sheaves on X parametrized by $\mathbb{R}$:

$$E_t \ := \ F_i(E)(-[t]D),$$

where $[t]$ is an integral part of t and $\alpha_{i-1} < t - [t] \le \alpha_i$, with the convention that $\alpha_0 = \alpha_l - 1$ and $\alpha_{l+1} = 1$. In particular, we have $E_{\alpha_i} = F_i(E)$ for $i \ = \ 1,\ 2,\ \dots,\ l+1$. If the underlying sheaf E is locally free, then E_* will be called a parabolic vector bundle.

Calculation of the second parabolic Chern class[3]:

2.2 Parabolic Chern class

Lemma 2.2.1 Consider a general parabolic bundle (E_*, F_*, α_*). Then we can compute the parabolic Chern classes of E_* using the following formula on X. Let us assume that $deg(F_i) = l_i$ with corresponding weights α_i and $r_i = rank\left(\frac{F_i}{F_{i-1}}\right)$. Then the first and second parabolic Chern classes of E_* are given by,

$$c_1^{par}(E_*) = c_1(E) + \left(\sum_{i=1}^{l} r_i\alpha_i\right) D \tag{2.2.1}$$

and

$$c_2^{par}(E_*) = c_2(E) + \sum_{i=1}^{l} r_i\alpha_i\ (c_1(E)\cdot D) - \sum_{i=1}^{l} \alpha_i\,(l_i - l_{i+1}) + \frac{1}{2}\left\{\left(\sum_{i=1}^{l} r_i\alpha_i\right)\cdot\left(\sum_{j=1}^{l} r_j\alpha_j\right) - \left(\sum_{i=1}^{l} r_i\alpha_i^2\right)\right\} D^2 \tag{2.2.2}$$

The study of Chern classes in families of flat bundles and its various generalizations are discussed in[6].

Using Riemann-Roch theorem we recall that the Euler Characteristic of torsion free sheaf E on a smooth projective surface can be given by,

$$\chi(E) = \frac{c_1(E)\cdot(c_1(E) - K_X)}{2} - c_2(E) + r\chi(O_X)$$

where r is the rank of the sheaf E. We recall the following formula for the reduced Hilbert polynomial of a torsion free sheaf W on a smooth projective surface X from[7].

$$p_W(n) = \frac{n^2H^2}{2} + \left(\frac{H\cdot c_1(W)}{r} - \frac{K_X\cdot H}{2}\right)n + \frac{\left(c_1(W)^2 - 2c_2(W) - c_1(W)\cdot K_X\right)}{2r} + \chi(O_X) \tag{2.2.3}$$

Using the above polynomial, we are going to obtain the formula of the reduced Hilbert polynomial for the rank 2 parabolic torsion free sheaf.

3 Results and Discussion

Let E_* be the parabolic coherent sheaf with the underlying coherent sheaf E on X.

The following theorem relates the Euler characteristic of the underlying sheaf E of E_* with the parabolic Chern classes of E. For computational reasons, we have assumed that the sheaf E_* is a direct sum of parabolic line bundles. The result is true for any rank 2 parabolic torsion-free sheaf with a parabolic structure.

Theorem 3.1: Let E_* be a parabolic sheaf, which is the direct sum of the line bundles $L_i,\ i = 1,2$. Let α_i be the weight associated with the line bundle L_i. The Euler characteristic of a parabolic sheaf $E_*(m)$ is given by

$$par_\chi(E_*(m)) = \chi(E(m)) + \left(\frac{D^2 + K_X\cdot D}{2} - D\cdot Hm\right)(2 - (\alpha_1 + \alpha_2)) - c_1(E)\cdot D + \alpha_1 L_2\cdot D + \alpha_2 L_1\cdot D \tag{3.1.1}$$

Proof. The parabolic filtration of E_* is given by,

$$E = L_1 \oplus L_2 \supset L_1 \oplus L_2(-D) \supset L_1(-D) \oplus L_2(-D)$$

together with the system of weights. We use the Definition 1.8 of[2] to see that,

$$par_\chi(E_*(m)) = \chi(E(-D)(m)) + \alpha_1 X\left(\frac{L_1 \oplus L_2}{L_1 \oplus L_2(-D)}(m)\right) + \alpha_2\chi\left(\frac{L_1 \oplus L_2(-D)}{L_1(-D) \oplus L_2(-D)}(m)\right)$$

Which implies,

$$par_\chi(E_*(m)) = \chi(E(-D)(m)) + \alpha_1[\chi((L_1 \oplus L_2)(m)) - \chi((L_1 \oplus L_2(-D))(m))] +$$
$$\alpha_2[\chi((L_1 \oplus L_2(-D))(m)) - \chi((L_1(-D) \oplus L_2(-D))(m))]$$

We use the formula for Euler Characteristic of torsion-free sheaves on surfaces (2.2.3) and using Problem 5.2(a), Chapter III of[8] to obtain:

$$\chi(E(-D)(m)) = 2\left\{\frac{m^2H^2}{2} + \left(\frac{H\cdot c_1(L_1(-D)\oplus L_2(-D))}{2} - \frac{K_X\cdot H}{2}\right)m \right.$$
$$\left. + \frac{(c_1(L_1(-D)\oplus L_2(-D)))^2 - 2c_1(L_1(-D))c_1(L_2(-D)) - c_1(L_1(-D)\oplus L_2(-D))\cdot K_X)}{4} + \chi(O_X)\right\}$$

Which implies,

$$\chi(E(-D)(m)) = 2\left\{\frac{m^2H^2}{2} + \left(\frac{H\cdot L_1(-D) + H\cdot L_2(-D)}{2} - \frac{K_X\cdot H}{2}\right)m \right.$$
$$\left. + \frac{\left(L_1(-D)^2 + L_2(-D)^2 - L_1(-D)\cdot K_X - L_2(-D)\cdot K_X\right)}{4} + \chi(O_X)\right\}$$

Which implies,

$$\chi(E(-D)(m)) = 2\left\{\frac{m^2H^2}{2} + \left(\frac{H\cdot L_1 - H\cdot D + H\cdot L_2 - H\cdot D}{2} - \frac{K_X\cdot H}{2}\right)m \right.$$
$$\left. + \frac{\left(L_1^2 - 2L_1\cdot D + D^2 + L_2^2 - 2L_2\cdot D + D^2 - L_1\cdot K_X + D\cdot K_X - L_2\cdot K_X + D\cdot K_X\right)}{4} + \chi(O_X)\right\}$$

Rearranging the terms, we get:

$$\chi(E(-D)(m)) = 2\left\{\frac{m^2H^2}{2} + \left(\frac{H\cdot L_1 + H\cdot L_2}{2} - \frac{K_X\cdot H}{2} - H\cdot D\right)m \right.$$
$$\left. + \frac{\left(L_1^2 - 2L_1\cdot D + L_2^2 - 2L_2\cdot D - L_1\cdot K_X - L_2\cdot K_X\right)}{4} + \frac{D^2 + D\cdot K_X}{2} + \chi(O_X)\right\}$$

Also,

$$\chi((L_1\oplus L_2)(m)) - \chi((L_1\oplus L_2(-D))(m)) = 2\left\{\frac{m^2H^2}{2} + \left(\frac{H\cdot L_1 + H\cdot L_2}{2} - \frac{K_X\cdot H}{2}\right)m + \frac{\left(L_1^2 + L_2^2 - L_1\cdot K_X - L_2\cdot K_X\right)}{4} + \chi(O_X)\right\}$$
$$-2\left\{\frac{m^2H^2}{2} + \left(\frac{H\cdot L_1 + H\cdot L_2(-D)}{2} - \frac{K_X\cdot H}{2}\right)m + \frac{\left(L_1^2 + L_2(-D)^2 - L_1\cdot K_X - L_2(-D)\cdot K_X\right)}{4} + \chi(O_X)\right\}$$

Implies,

$$\chi((L_1\oplus L_2)(m)) - \chi((L_1\oplus L_2(-D))(m)) = 2\left\{\left(\frac{H\cdot L_2 - H\cdot L_2(-D)}{2}\right)m + \frac{\left(L_2^2 - L_2(-D)^2 - L_2\cdot K_X + L_2(-D)\cdot K_X\right)}{4}\right\}$$

Thus,

$$\chi((L_1\oplus L_2)(m)) - \chi((L_1\oplus L_2(-D))(m)) = 2\left\{\left(\frac{H\cdot D}{2}\right)m + \frac{(2L_2\cdot D - D^2 - D\cdot K_X)}{4}\right\}$$

Similarly,

$$\chi((L_1\oplus L_2(-D))(m)) - \chi((L_1(-D)\oplus L_2(-D))(m)) = 2\left\{\left(\frac{H\cdot L_1 - H\cdot L_2(-D)}{2}\right)m + \frac{\left(L_2^2 - L_2(-D)^2 - L_2\cdot K_X + L_2(-D)\cdot K_X\right)}{4}\right\}$$

Thus,

$$\chi((L_1\oplus L_2(-D))(m)) - \chi((L_1(-D)\oplus L_2(-D))(m)) = 2\left\{\left(\frac{H\cdot D}{2}\right)m + \frac{(2L_1\cdot D - D^2 - D\cdot K_X)}{4}\right\}$$

Putting all these values in original equation, we get,

$$par_\chi(E_*(m)) = 2\left\{\tfrac{m^2H^2}{2} + \left(\tfrac{H\cdot L_1 + H\cdot L_2}{2} - \tfrac{K_X\cdot H}{2} - H\cdot D\right)m + \tfrac{\left(L_1^2 - 2L_1\cdot D + L_2^2 - 2L_2\cdot D - L_1\cdot K_X - L_2\cdot K_X\right)}{4} + \tfrac{D^2 + D\cdot K_X}{2} + \chi(O_X)\right\}$$
$$+2\alpha_1\left\{\left(\tfrac{H\cdot D}{2}\right)m + \tfrac{\left(2L_2\cdot D - D^2 - D\cdot K_X\right)}{4}\right\} + 2\alpha_2\left\{\left(\tfrac{H\cdot D}{2}\right)m + \tfrac{\left(2L_1\cdot D - D^2 - D\cdot K_X\right)}{4}\right\}$$

Rearranging the terms, we get:

$$par_\chi(E_*(m)) = 2\left\{\tfrac{m^2H^2}{2} + \left(\tfrac{H\cdot L_1 + H\cdot L_2}{2} - \tfrac{K_X\cdot H}{2}\right)m + \tfrac{\left(L_1^2 + L_2^2 - L_1\cdot K_X - L_2\cdot K_X\right)}{4} + \chi(O_X) - \tfrac{L_1\cdot D + L_2\cdot D}{2} + \tfrac{D^2 + D\cdot K_X}{2} - H\cdot D\right\}$$
$$+2\alpha_1\left\{\left(\tfrac{H\cdot D}{2}\right)m + \tfrac{L_2\cdot D}{2} - \tfrac{D^2 + D\cdot K_X}{4}\right\} + 2\alpha_2\left\{\left(\tfrac{H\cdot D}{2}\right)m + \tfrac{L_1\cdot D}{2} - \tfrac{D^2 + D\cdot K_X}{4}\right\}$$

Thus, further reducing, we obtain:

$$par_\chi(E_*(m)) = \chi(E(m)) + (2 - (\alpha_1 + \alpha_2))\left(\frac{D^2 + D\cdot K_X}{2} - (H\cdot D)m\right) - (1-\alpha_1)L_2\cdot D - (1-\alpha_2)L_1\cdot D.$$

Further, we rearrange the terms to get the formula stated in the theorem.

Theorem 3.2: Let F_* be any parabolic subbundle of rank 1 with weight α of E_*. The parabolic Hilbert polynomial of $F_*(m)$ is given by,

$$par_\chi(F_*(m)) = \chi(F(m)) + (1-\alpha)\left(\frac{D^2 + D\cdot K_X}{2} - (D\cdot H)m\right) - (1-\alpha)c_1(F)\cdot D$$

Proof. The Euler characteristic of the parabolic sheaf F_* using Definition 1.8 [2] is given by:

$$par_\chi(F_*(m)) = \chi(F(-D)(m)) + \alpha\chi\left(\tfrac{F}{F(-D)}(m)\right) = \chi(F(-D)(m))$$
$$+\alpha(\chi(F(m)) - \chi(F(-D)(m))) = (1-\alpha)(\chi(F(-D))(m)) + \alpha(\chi(F)(m))$$

Now we substitute the usual Euler characteristic of the torsion-free sheaf on surfaces, and we obtain:

$$par_\chi(F_*(m)) = (1-\alpha)\left\{\tfrac{m^2H^2}{2} + \left(\tfrac{H\cdot c_1(F(-D))}{1} - \tfrac{K_X\cdot H}{2}\right)m + \tfrac{(c_1(F(-D)))^2 - c_1(F(-D))\cdot K_X}{2} + \chi(O_X)\right\}$$
$$+\alpha\left\{\tfrac{m^2H^2}{2} + \left(\tfrac{H\cdot c_1(F)}{1} - \tfrac{K_X\cdot H}{2}\right)m + \tfrac{(c_1(F))^2 - c_1(F)\cdot K_X}{2} + \chi(O_X)\right\}$$

Which implies,

$$par_\chi(F_*(m)) = (1-\alpha)\left\{\tfrac{m^2H^2}{2} + \left(\tfrac{H\cdot F - H\cdot D}{1} - \tfrac{K_X\cdot H}{2}\right)m + \tfrac{F^2 - 2F\cdot D + D^2 - F\cdot K_X + D\cdot K_X}{2} + \chi(O_X)\right\}$$
$$+\alpha\left\{\tfrac{m^2H^2}{2} + \left(\tfrac{H\cdot F}{1} - \tfrac{K_X\cdot H}{2}\right)m + \tfrac{F^2 - F\cdot K_X}{2} + \chi(O_X)\right\}$$

Reducing further, we get:

$$par_\chi(F_*(m)) = \tfrac{m^2H^2}{2} + \left(\tfrac{F\cdot H - D\cdot H}{1} - \tfrac{K_X\cdot H}{2}\right)m + \tfrac{F^2 - 2(F\cdot D) + D^2 - F\cdot K_X + D\cdot K_X}{2} + \chi(O_X)$$
$$+\alpha\left((D\cdot H)m + \tfrac{2(F\cdot D) - D^2 - D\cdot K_X}{2}\right)$$

Which implies,

$$par_\chi(F_*(m)) = \tfrac{m^2H^2}{2} + \left(F\cdot H - \tfrac{K_X\cdot H}{2}\right)m + \tfrac{F^2 - F\cdot K_X}{2}$$
$$+\chi(O_X) - (1-\alpha)(D\cdot H)m + (1-\alpha)\tfrac{D\cdot K_X}{2} + (1-\alpha)\tfrac{D^2}{2} - (1-\alpha)(F\cdot D)$$

Further, we rearrange the terms to get the formula stated in the theorem.

Let us consider some examples.

If L is a line bundle and D is a Cartier divisor on X such that $L = O_X(D)$, then $c(L) = 1 + D$ (in other words, $c_1(L)$ is the class of D). (Notice that if D' is another choice for a divisor that gives L, then it is linearly equivalent to D, hence rationally equivalent, so $c_1(L)$ is well-defined.)

Example 3.1. Every line bundle on $\mathbb{P}^n$ is of the form $O_{\mathbb{P}^2}(a)$ for some $a \in \mathbb{Z}$, corresponding to the divisor aH; therefore, this line bundle has a total Chern class $1 + aH$. Hence, a direct sum of line bundles $O(a) \oplus O(b)$ *for some* $a, b \in \mathbb{Z}$,has a total Chern class $(1 + aH)(1 + bH) = 1 + (a + b)H + abH^2$.

Example 3.2. Let $X = \mathbb{P}^2$ and $E = O_{\mathbb{P}^2}(a) \oplus O_{\mathbb{P}^2}(b)$, with $a,\ b > 0$ be a rank 2 bundle on $\mathbb{P}^2$. Then $c(E) = (1 + aH)(1 + bH)$ (where $H = c_1(O_{\mathbb{P}^2}(a))$ is the class of a line bundle). We consider a parabolic structure for E as follows:

$$E = O_{\mathbb{P}^2}(a) \oplus O_{\mathbb{P}^2}(b) \supset O_{\mathbb{P}^2}(a) \oplus O_{\mathbb{P}^2}(-bD) \supset O_{\mathbb{P}^2}(-aD) \oplus O_{\mathbb{P}^2}(-bD) = E(-D)$$

together with the system of weights $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{2}{3}$ corresponding to $E_1 = O_{\mathbb{P}^2}(-aD) \oplus O_{\mathbb{P}^2}(-bD)$ and $E_2 = O_{\mathbb{P}^2}(a) \oplus O_{\mathbb{P}^2}(-bD)$ respectively. We have

$$E_1 = \{(s_1, s_2) \in E \mid s_1|_D = 0,\ s_2|_D = 0\}$$

$$\frac{E_2}{E_1} = \{(s_1, s_2) \in E \mid s_1|_D \neq 0,\ s_1|_D \neq 0\}$$

Clearly, $E(-D) = E_1$. Also, $c_1(E) = c_1(O_{\mathbb{P}^2}(a)) \oplus c_1(O_{\mathbb{P}^2}(b)) = (a+b)H$. And since we know the total Chern class $c(E)$, we also know that $c_2(E) = abH^2$.

Also, $K_{P^2} = -3H$.

Substituting these values in

$$par_\chi(E_*(m)) = \chi(E(m)) + \left(\frac{D^2 + K_X \cdot D}{2} - D \cdot Hm\right)(2 - (\alpha_1 + \alpha_2)) - c_1(E) \cdot D + \alpha_1 L_2 \cdot D + \alpha_2 L_1 \cdot D$$

We get,

$$par_\chi(E_*(m)) = 2\left\{\tfrac{m^2H^2}{2} + \left(\tfrac{H \cdot c_1(E)}{r} - \tfrac{K_X \cdot H}{2}\right)m + \tfrac{(c_1(E)^2 - 2c_2(E) - c_1(E) \cdot K_X}{2r} + \chi(O_X)\right\}$$
$$+ \left(\tfrac{D^2 - 3H \cdot D}{2} - D \cdot Hm\right)\left(2 - \left(\tfrac{1}{3} + \tfrac{2}{3}\right)\right) - aH \cdot D + \tfrac{1}{3}c_1(O_{P^2}(b)) \cdot D + \tfrac{2}{3}c_1(O_{P^2}(a)) \cdot D$$

$$par_\chi(E_*(m)) = 2\left\{\tfrac{m^2H^2}{2} + \left(\tfrac{H \cdot (a+b)H}{2} + \tfrac{3H^2}{2}\right)m + \left(\tfrac{(a+b)^2H^2 - 2abH^2 + 3(a+b)H^2}{2}\right) + 1\right\}$$
$$+ \left(\tfrac{D^2 - 3H \cdot D}{2} - D \cdot Hm\right) - (a+b)H \cdot D + \tfrac{1}{3}bH \cdot D + \tfrac{2}{3}aH \cdot D$$

$$par_\chi(E_*(m)) = m^2H^2 + (a+b+3)H^2m + \left(\frac{a^2 + b^2 + 3(a+b)}{2}\right)H^2 + 2 + \left(\frac{D^2 - 3H \cdot D}{2} - D \cdot Hm\right) - \frac{1}{3}aH \cdot D - \frac{2}{3}bH \cdot D$$

$$par_\chi(E_*(m)) = m^2 + (a+b+3)m - \left(m + \left(\frac{1}{3}a + \frac{2}{3}b + \frac{3}{2}\right)\right)H \cdot D + \left(\frac{a^2 + b^2 + 3(a+b)}{2}\right) + \frac{D^2}{2} + 2$$

$(Because\ H^2 = O_{\mathbb{P}^2}^2 = 1)$

If we consider a specific divisor D to be the line $z = 0$ in $\mathbb{P}^2$, then $D^2 = deg_D\left(N_{D/P^2}\right) = 1$ $(as\ N_{D/P^2} \sim O_{\mathbb{P}^2}(1))$. Also, $H \cdot D = 1$.

When we change these values in the equation above, we obtain:

$$par_\chi(E_*(m)) = m^2 + (a+b+3)m - m - \left(\frac{1}{3}a + \frac{2}{3}b + \frac{3}{2}\right) + \left(\frac{a^2 + b^2 + 3(a+b)}{2}\right) + \frac{1}{2} + 2$$

$$par_{\chi}\left(E_{*}\left(m\right)\right)=m^{2}+\left(a+b+2\right)m+\left(\frac{a^{2}+b^{2}}{2}+\frac{7a+5b}{6}\right)+1$$

Specifically, if we consider $E = O_{\mathbb{P}}\left(1\right)\oplus O_{\mathbb{P}^2}\left(1\right)$, we get:

$$par_{\chi}\left(E_{*}\left(m\right)\right)=m^{2}+4m+4$$

Using Equations (2.2.1) and (2.2.2), we can define the parabolic Chern classes for a rank 2 parabolic bundle as: Let E_* be a rank 2 parabolic bundle over X with underlying sheaf E. Now, as per the definition of the parabolic Chern classes, we have:

$$c_{1}^{par}\left(E_{*}\right)=c_{1}\left(E\right)+\left(\alpha_{1}+\ \alpha_{2}\right)D,$$

$$c_{2}^{par}\left(E_{*}\right)=c_{2}\left(E\right)+\left(\alpha_{1}+\ \alpha_{2}\right)\left(c_{1}\left(E\right)\cdot D\right)+\alpha_{1}\alpha_{2}D^{2}.$$

Thus, for the above example 3.2, we get,

$$c_{1}^{par}\left(E_{*}\right)=\left(a+b\right)H+D$$

$$c_{2}^{par}\left(E_{*}\right)=abH^{2}+\left(a+b\right)\left(H\cdot D\right)+\frac{2}{9}D^{2}$$

4 Conclusion

In summary, the paper achieves its objectives by providing a comprehensive analysis of parabolic rank 2 bundles and offering a detailed computation of the parabolic Hilbert polynomial. These contributions significantly enrich the existing body of knowledge in the field of algebraic geometry and pave the way for future research avenues.

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