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RESEARCH ARTICLE

Characterizations of the Direct Sum of Two Difference - Mean Fuzzy Graphs

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Abstract

Objectives: This study presents a new type of fuzzy graph known as the difference mean fuzzy graph by introducing difference mean edge. **Methodology:** In this paper, difference mean edge in a fuzzy graph is defined by considering the relationship between the membership value of the edge and the membership values of its end vertices. Also, difference mean fuzzy graph is defined and its properties are derived. **Findings:** The difference mean edge and the difference mean fuzzy graph are introduced. The requirements for an edge in the direct sum of two fuzzy graphs to be a difference mean edge are found in this study. Additionally, conditions are derived such that the direct sum of two fuzzy graphs is a difference mean fuzzy graph. **Novelty:** Depending on the membership values of the edges and vertices, effective edge in fuzzy graph have already been defined. A new concept of difference mean edge in fuzzy graph is introduced. Using this, difference mean fuzzy graph is also introduced. Characterizations of the difference mean edge in the direct sum of fuzzy graphs are attained. The requirements for the necessary and sufficient component of difference mean fuzzy graphs to be a direct sum are suggested.

Mathematics Subject Classification (2020): 05C72**,** 05C76.

Keywords: Difference mean edge; Difference Mean fuzzy graph; Effective fuzzy graph; Effective difference mean edge; Direct sum

1 Introduction

A relatively new mathematical framework called a fuzzy set is being used to illustrate the idea of uncertainty in everyday struggles. It was first introduced by Zadeh in 1965 [(1)], but the ideas were first explored by a number of separate studies in the 1970s, including those by Rosenfeld, Bhutani and Battou. They investigated several fuzzy analogs of graph theoretic notions, including paths, cycles, and connectedness. Fuzzy graphs have the potential to tackle a wide range of issues. Despite being extremely new, it has rapidly expanded and has many uses across many industries. Fuzzy graph research has been growing exponentially in the field of mathematics as well as in its applications in science and technology. S. R. Latha and Nagoorgani investigated a few properties about the operations of fuzzy graphs in 2014. The features of a fuzzy labeling graph and a comment on fuzzy labeling were examined by A. Nagoor Gani and D. Rajalaxmi in 2012. A. Elumalai conducted an analysis of the latest trends in fuzzy graphs[2]. A new kind of fuzzy graph is provided in this paper, called the difference mean fuzzy graph. By placing a condition on the membership value of the edges, effective edge and effective fuzzy graph have already been defined. Here the difference mean edge and difference mean fuzzy graphs are proposed, and their characteristics are analyzed. They will be helpful in analyzing situations that can be modeled into fuzzy graphs. Effective, complete, and regular features of the direct sum of two fuzzy graphs have been researched since its introduction by K. Radha and S. Arumugam in 2013. T. Henson and Santhanalakshmi explored the widened form of direct sum of fuzzy graphs[3], whereas Nagadurga and Subhashish looked at current trends in fuzzy graph properties in fuzzy operations and applications[4]. K. Radha and P. Indumathi provided fundamental knowledge regarding the characteristics of the direct sum in fuzzy graphs[5]. The qualities of difference mean edges in direct sum and their relationship to the properties of the direct sum of two difference mean fuzzy graphs are the focus of our research in this work. Additionally, the requirements for the necessary and sufficient component of difference mean fuzzy graphs to be a direct sum are suggested.

Now we can remember some of the basic denotations that we employed for our primary findings in this section.

Let $G:(\sigma,\mu)$ be a fuzzy graph on a graph $G^*:(V,E)$ is a pair of functions $\sigma:V\to[0,1]$ and $\mu:V\times V\to[0,1]$, in such a way that, $\mathrm{m}(a,b)\leq\sigma(a)\sigma(b)$,$\forall a,b\in V$, then $H:(\tau,\rho)$ is considered a fuzzy subgraph of $G:(\sigma,\mu)$ if $\tau(a)\leq\sigma(a)$, $\forall a\in V(H)$ and $\rho(ab)\leq\mu(ab)$, $\forall ab\in E(H)$. If $\mu(ab)=\sigma(a)\wedge\sigma(b)$, $\forall ab\in E$, then the fuzzy graph is effective[6].

$G_1:(\sigma_1,\mu_1)$ and $G_2:(\sigma_2,\mu_2)$ represent two fuzzy graphs with corresponding crisp graphs $G_1{}^*:(V_1,E_1)$ and $G_2{}^*:(V_2,E_2)$. If, V= $V_1\cup V_2$, $E=\{uv/u,v\in V;\ uv\in E_1\ or\ uv\in E_2\ but\ not\ both\}$. $G:(\sigma,\mu)$ can be defined by:

$$\sigma(u)=\begin{cases}\sigma_1(u)\ ;\ if\ u\varepsilon V_1\\ \sigma_2(u)\ ;\ if\ u\varepsilon V_2\\ \sigma_1(u)\vee\sigma_2(u);\ if\ u\varepsilon V_1\cup V_2\end{cases}$$

$$\mu(uv)=\begin{cases}\mu_1(uv);\ if\ uv\varepsilon E_1\\ \mu_2(uv);\ if\ uv\varepsilon E_2\end{cases}$$

Then, if $uv\in E_1$, $\mu(uv)=\mu_1(uv)\leq\sigma_1(u)\wedge\sigma_1(v)\leq\sigma(u)\wedge\sigma(v)$, if $uv\in E_2$, $\mu(uv)=\mu_2(uv)\leq\sigma_2(u)\wedge\sigma_2(v)\leq\sigma(u)\wedge\sigma(v)$.. Therefore, (σ,μ) specifies a fuzzy graph. We refer to this as the direct sum of two fuzzy graphs.[7] (K. Radha and S. Arumugam May 2013)

2 Methodology

Depending on the membership values of the edges and vertices, effective edge in fuzzy graph have already been defined. In this paper, difference mean edge in a fuzzy graph is defined by considering the relationship between the membership value of the edge and the membership values of its end vertices. Also, difference mean fuzzy graph is defined and its properties are derived.

2.1: Definition: Consider a fuzzy graph $G:(\sigma,\mu)$ on $G^*:(V,E)$. Let z_1z_2 be an edge in G. If $\mu(z_1,z_2)=\frac{|\sigma(z_1)-\sigma(z_2)|}{2}$ or if $2\mu(z_1,z_2)=|\sigma(z_1)-\sigma(z_2)|$, then z_1z_2 is a difference mean edge. If every edge in G is a difference mean edge, then G is a difference mean fuzzy graph.

For example, the following Figure 1 is a difference mean fuzzy graph.

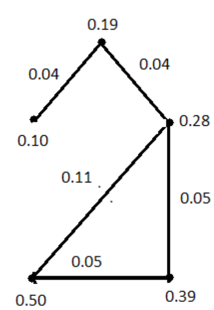

Fig 1. Example for a Difference mean fuzzy graph

3 Results and Discussions

3.1 Difference Mean Edge in Direct Sum

One of the essential ideas in graph and fuzzy graph theory, the ring sum of fuzzy graphs has several applications in various domains. In order to analyze aspects that arise from the interaction of the two fuzzy graphs, it can be helpful to combine two fuzzy graphs using the ring sum operation. It is especially helpful for studying the circuits inside of graphs. Applications for ring sums can be found in computer science in fields such as data mining, image processing, and computer networks. They are also employed in the expression of arbitrary circuits as a linear combination of basic circuits and in structural analysis. In this section, the difference mean edge property in the direct sum of two fuzzy graphs is derived.

Remarks:

The difference mean edge property of an edge in fuzzy graph G_1 or G_2 need not be preserved in the direct sum $G_1 \oplus G_2$. For example, consider the edge uv in G_1 of Figure 2.

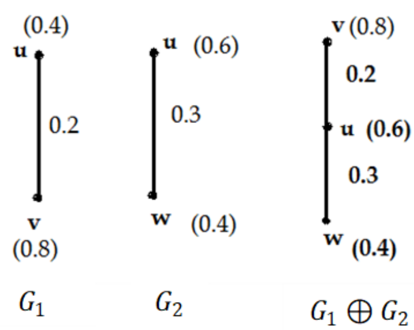

Fig 2. The difference mean edge property of an edge in fuzzy graphG_1 or G_2 need not be preserved in the direct sum $G_1 \oplus G_2$

Here $2\mu_1(uv) = 2 \times 0.2 = 0.4$ and $|\sigma_1(u) - \sigma_1(v)| = |0.4 - 0.8| = 0.4$

Therefore, $2\mu_1(uv) = |\sigma_1(u) - \sigma_1(v)|$. Hence, uv is a difference mean edge in G_1. But $2(\mu_1 \oplus \mu_2)(uv) = 2 \times 0.2 = 0.4 \neq 0.2 = |0.6 - 0.8| = |\sigma_1 \oplus \sigma_2(u) - \sigma_1 \oplus \sigma_2(v)|$ which implies that uv is not a difference mean edge in $G_1 \oplus G_2$.

An edge can be a difference mean edge in the direct sum $G_1 \oplus G_2$ without being a difference mean edge in G_1 or in G_2. For example, in Figure 3:

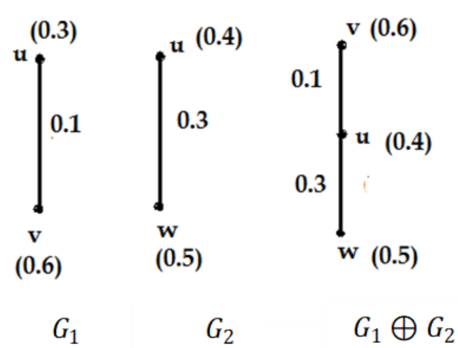

Fig 3. An edge can be a difference mean edge in the direct sum$G_1 \oplus G_2$ without being a difference mean edge in G_1 or in G_2

Here $2\mu_1(uv) = 2 \times 0.1 = 0.2 \neq 0.3 = |0.3 - 0.6| = |\sigma_1(u) - \sigma_1(v)|$ shows that uv is not a difference mean edge in G_1.

But $2(\mu_1 \oplus \mu_2)(uv) = 2 \times 0.1 = 0.2 = |0.4 - 0.6| = |\sigma_1 \oplus \sigma_2(u) - \sigma_1 \oplus \sigma_2(v)|$ implies that uv is a difference mean edge in $G_1 \oplus G_2$.

Characterizations and requirements for an edge to be a difference mean edge in the direct sum of two fuzzy graphs are obtained in the following theorems:

Theorem 3.1.1:

Let $G_1 : (V_1, E_1; \sigma_1, \mu_1)$ and $G_2 : (V_2, E_2; \sigma_2, \mu_2)$ be two fuzzy graphs. Let z_1z_2 be an edge in G_1 such that $z_1, z_2 \in V_1 - V_2$. Then z_1z_2 is a difference mean edge in the direct sum $G_1 \oplus G_2$ if and only if z_1z_2 is a difference mean edge in G_1.

Proof:

Since both z_1 and z_2 are in $V_1 - V_2$, the edge is in E_1 but not in E_2. So $z_1z_2 \in E_1 - E_2$. Therefore, by the definition of the direct sum,

$$(\sigma_1 \oplus \sigma_2)(z_1) = \sigma_1(z_1), (\sigma_1 \oplus \sigma_2)(z_2) = \sigma_1(z_2) \text{ and } (\mu_1 \oplus \mu_2)(z_1z_2) = \mu_1(z_1z_2) \tag{3.1}$$

Assume that z_1z_2 is a difference mean edge in the direct sum $G_1 \oplus G_2$. Then by the definition of the difference mean edge, $2(\mu_1 \oplus \mu_2)(z_1z_2) = |\sigma_1 \oplus \sigma_2(z_1) - \sigma_1 \oplus \sigma_2(z_2)|$ which implies that $2\mu_1(z_1z_2) = |\sigma_1(z_1) - \sigma_1(z_2)|$ by using (Equation (3.1)). Hence, z_1z_2 is a difference mean edge in G_1.

Conversely assume that z_1z_2 is a difference mean edge in G_1. Then by the definition of the difference mean edge, $2\mu_1(z_1z_2) = |\sigma_1(z_1) - \sigma_1(z_2)|$. Using (Equation (3.1)), this relation gives $2(\mu_1 \oplus \mu_2)(z_1z_2) = |\sigma_1 \oplus \sigma_2(z_1) - \sigma_1 \oplus \sigma_2(z_2)|$. Hence, z_1z_2 is a difference mean edge in the direct sum $G_1 \oplus G_2$.

Theorem 3.1.2:

If $G_1 : (V_1, E_1; \sigma_1, \mu_1)$ and $G_2 : (V_2, E_2; \sigma_2, \mu_2)$ be two fuzzy graphs. Let z_1z_2 be an edge in G_1 such that $z_1 \in V_1 \cap V_2$, $z_2 \in V_1 - V_2$ and $\sigma_1(z_1) \geq \sigma_2(z_1)$. Then z_1z_2 is a difference mean edge in the direct sum $G_1 \oplus G_2$ if and only if z_1z_2 is a difference mean edge in G_1.

Proof:

Since $z_1 \in V_1 \cap V_2$ and $z_2 \in V_1 - V_2$, the edge z_1z_2 cannot be in E_2. So $z_1z_2 \in E_1 - E_2$. Also, $\sigma_1(z_1) \geq \sigma_2(z_1)$. Therefore, since $z_1 \in V_1 \cap V_2$ and $\sigma_1(z_1) \geq \sigma_2(z_1)$, by the definition of the direct sum, $(\sigma_1 \oplus \sigma_2)(z_1) = \sigma_1(z_1) \vee \sigma_2(z_1) = \sigma_1(z_1)$. Also since $z_2 \in V_1 - V_2$ and $z_1z_2 \in E_1 - E_2$, by the definition of the direct sum, $(\sigma_1 \oplus \sigma_2)(z_2) = \sigma_1(z_2)$ and $(\mu_1 \oplus \mu_2)(z_1z_2) = \mu_1(z_1z_2)$.

Assume that z_1z_2 is a difference mean edge in the direct sum $G_1 \oplus G_2$. Then by the definition of the difference mean edge, $2(\mu_1 \oplus \mu_2)(z_1z_2) = |\sigma_1 \oplus \sigma_2(z_1) - \sigma_1 \oplus \sigma_2(z_2)|$ which implies that $2\mu_1(z_1z_2) = |\sigma_1(z_1) - \sigma_1(z_2)|$ by using (Equation (3.1)). Hence, z_1z_2 is a difference mean edge in G_1.

Conversely, assume that z_1z_2 is a difference mean edge in G_1. Then by the definition of the difference mean edge, $2\mu_1(z_1z_2) = |\sigma_1(z_1) - \sigma_1(z_2)|$. Using (Equation (3.1)), this relation gives $2(\mu_1 \oplus \mu_2)(z_1z_2) = |\sigma_1 \oplus \sigma_2(z_1) - \sigma_1 \oplus \sigma_2(z_2)|$. Hence, z_1z_2 is a difference mean edge in the direct sum $G_1 \oplus G_2$.

Theorem 3.1.3:

Let $G_1 : (V_1, E_1; \sigma_1, \mu_1)$ and $G_2 : (V_2, E_2; \sigma_2, \mu_2)$ be two fuzzy graphs. Let z_1z_2 be an edge in G_1 such that $z_1z_2 \in E_1 - E_2$; $z_1, z_2 \in V_1 \cap V_2$ with $\sigma_1(z_1) \geq \sigma_2(z_1)$ and $\sigma_1(z_2) \geq \sigma_2(z_2)$. Then z_1z_2 is an effective edge in the direct sum $G_1 \oplus G_2$ if and only if z_1z_2 is effective in G_1.

Proof:

Since, $z_1z_2 \in E_1 - E_2$, the edge z_1z_2 appears in $G_1 \oplus G_2$. Since $\sigma_1(z_1) \geq \sigma_2(z_1)$ and $\sigma_1(z_2) \geq \sigma_2(z_2)$, by the definition of the direct sum, $\sigma_1 \oplus \sigma_2(z_1) = \sigma_1(z_1) \vee \sigma_2(z_1) = \sigma_1(z_1)$ and $\sigma_1 \oplus \sigma_2(z_2) = \sigma_1(z_2) \vee \sigma_2(z_2) = \sigma_1(z_2)$. Also, $(\mu_1 \oplus \mu_2)(z_1z_2) = \mu_1(z_1z_2)$ since, $z_1z_2 \in E_1 - E_2$. Hence, z_1z_2 is a difference mean edge in the direct sum $G_1 \oplus G_2$

$\Leftrightarrow 2(\mu_1 \oplus \mu_2)(z_1z_2) = |\sigma_1 \oplus \sigma_2(z_1) - \sigma_1 \oplus \sigma_2(z_2)| \Leftrightarrow 2\mu_1(z_1z_2) = |\sigma_1(z_1) - \sigma_1(z_2)|$

$\Leftrightarrow$ z_1z_2 is a difference mean edge in G_1.

Theorem 3.1.4:

Let $G_1 : (V_1, E_1; \sigma_1, \mu_1)$ and $G_2 : (V_2, E_2; \sigma_2, \mu_2)$ be two fuzzy graphs. If z_1z_2 is an edge in G_1 such that $z_1z_2 \in E_1 - E_2$; $z_1, z_2 \in V_1 \cap V_2$ with $|\sigma_1(z_1) \vee \sigma_2(z_1) - \sigma_1(z_2) \vee \sigma_2(z_2)| = 2\mu_1(z_1z_2)$, then z_1z_2 is a difference mean edge in the direct sum $G_1 \oplus G_2$.

Proof:

Since, $z_1, z_2 \in V_1 \cap V_2$, by the definition of the direct sum $\sigma_1 \oplus \sigma_2(z_1) = \sigma_1(z_1) \vee \sigma_2(z_1), \sigma_1 \oplus \sigma_2(z_2) = \sigma_1(z_2) \vee \sigma_2(z_2)$ and $(\mu_1 \oplus \mu_2)(z_1z_2) = \mu_1(z_1z_2)$. Therefore,

$|\sigma_1(z_1) \vee \sigma_2(z_1) - \sigma_1(z_2) \vee \sigma_2(z_2)| = 2\mu_1(z_1z_2)$

$\Rightarrow |\sigma_1 \oplus \sigma_2(z_1) - \sigma_1 \oplus \sigma_2(z_2)| = 2(\mu_1 \oplus \mu_2)(z_1z_2).$

Hence, z_1z_2 is a difference mean edge in $G_1 \oplus G_2$.

3.2 Difference- mean fuzzy graphs in direct sum

Remarks:

The direct sum of difference mean fuzzy graphs need not be a difference mean fuzzy graph. For example, in Figure 4:

Here, both G_1 and G_2 are difference mean fuzzy graphs. But their direct sum $G_1 \oplus G_2$ is not a difference mean fuzzy graph.

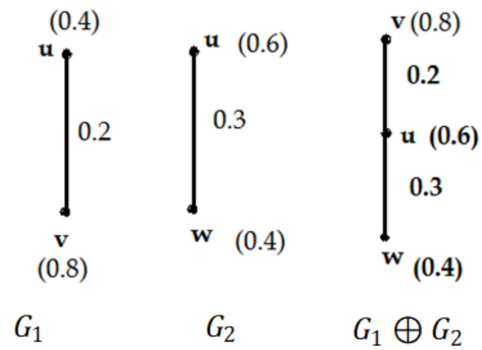

Fig 4. The direct sum of difference mean fuzzy graphs need not be a difference mean fuzzy graph

$G_1 \oplus G_2$ can be a difference mean fuzzy graph, when G_1 or G_2 is not a difference mean fuzzy graph. For example, the fuzzy graph G_2 in Figure 5, is not a difference mean fuzzy graph. But the direct sum $G_1 \oplus G_2$ is a difference mean fuzzy graph.

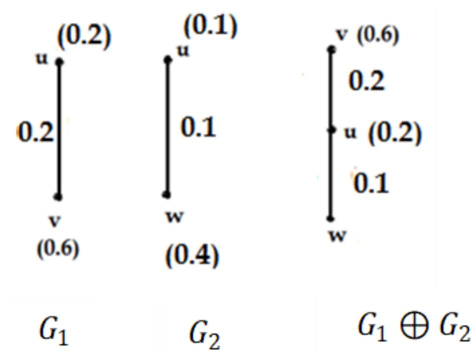

Fig 5. $G_1 \oplus G_2$ can be a difference mean fuzzy graph, when G_1 or G_2 is not a difference mean fuzzy graph

The requirements for the direct sum of two fuzzy graphs to be a difference mean fuzzy graph are found in the ensuing theorems.

Theorem 3.2.1:

Let $G_1 : (V_1, E_1; \sigma_1, \mu_1)$ and $G_2 : (V_2, E_2; \sigma_2, \mu_2)$ be two fuzzy graphs such that $V_1 \cap V_2 = \varnothing$. Then, the direct sum $G_1 \oplus G_2$ is a difference mean fuzzy graph if and only if, both G_1 and G_2 are difference mean fuzzy graphs.

Proof:

Since, $V_1 \cap V_2 = \varnothing$, $E_1 \cap E_2$ is also empty. Therefore, $G_1 \oplus G_2$ is a disconnected fuzzy graphs with two components G_1 and G_2. Also, $\sigma_1 \oplus \sigma_2(z_1) = \sigma_i(z_1)$ if $z_1 \in V_i, i = 1, 2$ and $(\mu_1 \oplus \mu_2)(a) = \mu_i(a)$, if $a \in E_i, i = 1, 2$. Therefore, for any edge, $z_1 z_2 \in E_i, i = 1,\ 2$, $2(\mu_1 \oplus \mu_2)(z_1 z_2) = |\sigma_1 \oplus \sigma_2(z_1) - \sigma_1 \oplus \sigma_2(z_2)| \Leftrightarrow 2\mu_i(z_1 z_2) = |\sigma_i(z_1) - \sigma_i(z_2)|$. Hence, $G_1 \oplus G_2$ is a difference mean fuzzy graph if and only if both G_1 and G_2 are difference mean fuzzy graphs.

Theorem 3.2.2:

If $G_1 : (V_1, E_1; \sigma_1, \mu_1)$ and $G_2 : (V_2, E_2; \sigma_2, \mu_2)$ be two fuzzy graphs such that $\sigma_1(z_1) = \sigma_2(z_1)$ for every, $z_1 \in V_1 \cap V_2$ and $E_1 \cap E_2 = \varnothing$. Then, the direct sum $G_1 \oplus G_2$ is a difference mean fuzzy graph if and only if G_1 and G_2 are difference mean fuzzy graphs.

Proof:

Since, $E_1 \cap E_2 = \varnothing$, all the edges of G_1 and G_2 appear in the direct sum $G_1 \oplus G_2$ and for any edge $z_1 z_2$ of G_i, both z_1 and z_2 cannot be in $V_1 \cap V_2$. Assume that $G_1 \oplus G_2$ is a difference mean fuzzy graph. For any edge $z_1 z_2$ of G_1, there are two cases to consider:

i) $z_1 \in V_1 \cap V_2 . z_2 \in V_i$ (or vice versa)

ii) $z_1, z_2 \in V_1$

Case (i): $z_1 \in V_1 \cap V_2, z_2 \in V_i$

Then, $\sigma_1 \oplus \sigma_2(z_1) = \sigma_1(z_1) \vee \sigma_2(z_1) = \sigma_1(z_1), \sigma_1 \oplus \sigma_2(z_2) = \sigma_1(z_2)$ since, $\sigma_1(z_1) = \sigma_2(z_1)$ and $(\mu_1 \oplus \mu_2)(z_1z_2) = \mu_1(z_1z_2)$ since, $G_1 \oplus G_2$ is difference mean fuzzy graph, $2(\mu_1 \oplus \mu_2)(z_1z_2) = |\sigma_1 \oplus \sigma_2(z_1) - \sigma_1 \oplus \sigma_2(z_2)| \Rightarrow 2\mu_1(z_1z_2) = |\sigma_1(z_1) - \sigma_1(z_2)| \Rightarrow z_1z_2$ is a difference mean edge in G_1.

Case (ii): $z_1, z_2 \in V_1$

Then $\sigma_1 \oplus \sigma_2(z_1) = \sigma_1(z_1), \sigma_1 \oplus \sigma_2(z_2) = \sigma_1(z_2)$ and $(\mu_1 \oplus \mu_2)(z_1z_2) = \mu_1(z_1z_2)$.

Therefore, proceeding as in case (i), z_1z_2 is a difference mean edge in G_1. Since z_1z_2 is an arbitrary edge, G_1 is a difference mean fuzzy graph. Similarly, G_2 is a difference mean fuzzy graph.

Conversely, assume that G_1 and G_2 are difference mean fuzzy graphs. Let z_1z_2 be any edge of $G_1 \oplus G_2$. Then, either $z_1z_2 \in E_1 - E_2$ or $z_1z_2 \in E_2 - E_1$. Without loss of generality, assume that $z_1z_2 \in E_1 - E_2$. Then, either $z_1 \in V_1 \cap V_2, z_2 \in V_i$ (or vice versa) or $z_1, z_2 \in V_1$. In both cases, $\sigma_1 \oplus \sigma_2(z_1) = \sigma_1(z_1), \sigma_1 \oplus \sigma_2(z_2) = \sigma_1(z_2)$ and $(\mu_1 \oplus \mu_2)(z_1z_2) = \mu_1(z_1z_2)$. Since, G_1 is a difference mean fuzzy graph, $2\mu_1(z_1z_2) = |\sigma_1(z_1) - \sigma_1(z_2)| \Rightarrow 2(\mu_1 \oplus \mu_2)(z_1z_2) = |\sigma_1 \oplus \sigma_2(z_1) - \sigma_1 \oplus \sigma_2(z_2)| \Rightarrow z_1z_2$ is a difference mean edge in $G_1 \oplus G_2$. The proof is similar, if $z_1z_2 \in E_2 - E_1$. Hence, $G_1 \oplus G_2$ is a difference mean fuzzy graph.

Theorem 3.2.3:

Let $G_1 : (V_1, E_1; \sigma_1, \mu_1)$ and $G_2 : (V_2, E_2; \sigma_2, \mu_2)$ be two fuzzy graphs such that $\sigma_1(z_1) = \sigma_2(z_1)$ for every $z_1 \in V_1 \cap V_2$ and $E_1 \cap E_2 \neq \varnothing$. If G_1 and G_2 are difference mean fuzzy graphs, then the direct sum $G_1 \oplus G_2$ is a difference mean fuzzy graph.

Proof:

Let z_1z_2 be any edge of $G_1 \oplus G_2$. Then $z_1z_2 \in E_1 - E_2$ or $z_1z_2 \in E_2 - E_1$

Case (i): $z_1z_2 \in E_1 - E_2$

Since, $z_1z_2 \notin E_2$, at most one of the vertices z_1, z_2 can be in $V_1 \cap V_2$.

The hypothesis of the statement is that $\sigma_1(z_1) = \sigma_2(z_1)$ for every $z_1 \in V_1 \cap V_2$.

Subcase 1.1: $z_1 \in V_1, z_2 \in V_1 \cap V_2$ (or vice versa)

Then, $\sigma_1(z_2) = \sigma_2(z_2)$ by hypothesis. Therefore, by the definition of the direct sum, $\sigma_1 \oplus \sigma_2(z_1) = \sigma_1(z_1), \sigma_1 \oplus \sigma_2(z_2) = \sigma_1(z_2)$ and, $(\mu_1 \oplus \mu_2)(z_1z_2) = \mu_1(z_1z_2)$.

Hence, z_1z_2 is a difference mean edge in $G_1 \Rightarrow 2\mu_1(z_1z_2) = |\sigma_1(z_1) - \sigma_1(z_2)| \Rightarrow 2(\mu_1 \oplus \mu_2)(z_1z_2) = |\sigma_1 \oplus \sigma_2(z_1) - \sigma_1 \oplus \sigma_2(z_2)| \Rightarrow z_1z_2$ is a difference mean edge in G_1. Therefore, the direct sum $G_1 \oplus G_2$ is a difference mean fuzzy graph.

Subcase 1.2: $z_1, z_2 \in V_1$

Then $\sigma_1 \oplus \sigma_2(z_1) = \sigma_1(z_1), \sigma_1 \oplus \sigma_2(z_2) = \sigma_1(z_2)$ and $(\mu_1 \oplus \mu_2)(z_1z_2) = \mu_1(z_1z_2)$, since $\sigma_1(z_2) = \sigma_2(z_2)$. Therefore, proceeding as in case i), z_1z_2 is a difference mean edge in G_1. Hence, the direct sum $G_1 \oplus G_2$ is a difference mean fuzzy graph.

Case (ii): $z_1z_2 \in E_2 - E_1$

The proof is similar to case (i), then the direct sum $G_1 \oplus G_2$ is a difference mean fuzzy graph.

Figure 6 is the example for the above theorem.

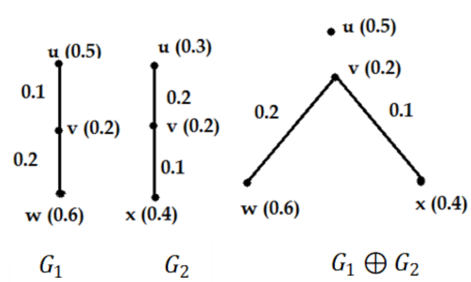

Fig 6. If $\sigma_1(z_1) = \sigma_2(z_1)$ for every $z_1 \in V_1 \cap V_2$ and $E_1 \cap E_2 \neq \varnothing$. If G_1 and G_2 are difference mean fuzzy graphs, then the direct sum $G_1 \oplus G_2$ is a difference mean fuzzy graph

Theorem 3.2.4:

Let $G_1 : (V_1, E_1; \sigma_1, \mu_1)$ and $G_2 : (V_2, E_2; \sigma_2, \mu_2)$ be two fuzzy graphs such that $\sigma_1(z_1) = \sigma_2(z_1)$ for every $z_1 \in V_1 \cap V_2$ and $E_1 \cap E_2 \neq \varnothing$. Then the direct sum $G_1 \oplus G_2$ is a difference mean fuzzy graph if and only if all the edges in $E_1 - E_2$ and in $E_2 - E_1$

are difference mean edges.

Proof:

Assume that $G_1 \oplus G_2$ is a difference mean fuzzy graph.

Let $z_1z_2 \in E_1 - E_2$ be any element. Then z_1z_2 appears in the direct sum $G_1 \oplus G_2$ and $(\mu_1 \oplus \mu_2)(z_1z_2) = \mu_1(z_1z_2)$.

Since, $z_1z_2 \notin E_2$, both z_1 and z_2 cannot be in V_2. If $z_1 \in V_1$ and $z_2 \in V_1 \cap V_2$, then $\sigma_1 \oplus \sigma_2(z_1) = \sigma_1(z_1), \sigma_1 \oplus \sigma_2(z_2) = \sigma_1(z_2) \vee \sigma_2(z_2) = \sigma_1(z_2)$, since $\sigma_1(z_2) = \sigma_2(z_2)$. Since, z_1z_2 is a difference mean edge in $G_1 \oplus G_2$,

$\Rightarrow 2(\mu_1 \oplus \mu_2)(z_1z_2) = |\sigma_1 \oplus \sigma_2(z_1) - \sigma_1 \oplus \sigma_2(z_2)| \Rightarrow 2\mu_1(z_1z_2) = |\sigma_1(z_1) - \sigma_1(z_2)|$

$\Rightarrow z_1z_2$ is a difference mean edge in G_1. The proof is similar, if $z_1 \in V_1 \cap V_2$ and $z_2 \in V_1$. If both $z_1, z_2 \in V_1$, then by the definition of the direct sum and since $G_1 \oplus G_2$ is a difference mean fuzzy graph,

$\Rightarrow 2(\mu_1 \oplus \mu_2)(z_1z_2) = |\sigma_1 \oplus \sigma_2(z_1) - \sigma_1 \oplus \sigma_2(z_2)| \Rightarrow 2\mu_1(z_1z_2) = |\sigma_1(z_1) - \sigma_1(z_2)|$

$\Rightarrow z_1z_2$ is a difference mean edge in G_1. Since, z_1z_2 is arbitrary, all the elements of $E_1 - E_2$ are difference mean edges in G_1.

Similarly, all the elements of $E_2 - E_1$ are difference mean edges in G_2.

Conversely, assume that all the elements in $E_1 - E_2$ and in $E_2 - E_1$ are difference mean edges.

Let z_1z_2 be any edge of $G_1 \oplus G_2$. Then $z_1z_2 \in E_1 - E_2$ or $z_1z_2 \in E_2 - E_1$. Assume that $z_1z_2 \in E_1 - E_2$. Then, z_1z_2 is a difference mean edge in G_1.

$\Rightarrow 2\mu_1(z_1z_2) = |\sigma_1(z_1) - \sigma_1(z_2)| \Rightarrow 2(\mu_1 \oplus \mu_2)(z_1z_2) = |\sigma_1 \oplus \sigma_2(z_1) - \sigma_1 \oplus \sigma_2(z_2)|$

$\Rightarrow z_1z_2$ is a difference mean edge of $G_1 \oplus G_2$.

The proof is similar, if $z_1z_2 \in E_2 - E_1$. Hence, $G_1 \oplus G_2$ is a difference mean fuzzy graph.

4 Conclusion

A contribution towards the classification of edges in fuzzy graphs has been made in this paper by introducing the difference mean edge. Also, this paper introduces difference-mean fuzzy graphs and discusses some of their features. We derive the requirements for the direct sum of two fuzzy graphs to be a difference mean fuzzy graph. Furthermore, requirements are derived for an edge in the direct sum of two fuzzy graphs to have a difference mean edge. This attempt to introduce a new edge in fuzzy graphs will be helpful in studying and analyzing situations that can be modeled into fuzzy graphs. It will be useful to introduce and examine particular edges in fuzzy graphs based on their membership values when investigating the properties of graphs that result from relationships between items in everyday life. Properties of difference mean edges in other operations of fuzzy graphs like union, intersection, join, composition, etc. can be studied in the future.

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