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RESEARCH ARTICLE

Fractional Star Domination Number of Graphs

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Abstract

Objectives: Consider a graph $G(V,E)$is the connected, undirected graph. This study creates a new parameter named the Fractional Star Domination Number (FRSDN), which is denoted by $\gamma_{gs}(G)$ to calculate the minimum weight with the star of all vertices of G and the function values of all edges. **Methods:** This study evaluates the $\gamma_{gs}(G)$ of some standard graphs and bounds by generalizing the value that is provided by the function value of an edges. **Findings:** This study evaluated $\gamma_{gs}(G)$ on some standard graphs, such as paths, cycles, and the rooted product of paths and cycles. Finally, we obtain some bounds on $\gamma_{gs}(G)$ for some general graphs, as well as the exactness value yielded by them. For any graph G without isolated vertices, we have $\gamma_{gs}(G) \geq \gamma_{f'}(G)$. **Novelty:** The new parameter FRSDN of G is created by combining the fractional dominating function and the star dominating set.

Keywords: Domination Number; Star Domination Number; Fractional Dominating Function; Fractional Domination Number; Fractional Star Domination Number

1 Introduction

A Graph G has vertex set $V(G)$ and an edge set $E(G)$ and is an undirected connected graph. The open neighbourhood of an edge q, which is defined as the collection of all edges adjacent to q and is denoted by $Nbhd(q)$. The closed neighbourhood of an edge is defined as follows $Nbhd[q] = Nbhd(q) \cup \{q\}$. The star of r is the gathering of all edges that are incident to the vertex r, and it is notated by $S^*(r)$. Here, the degree of the vertex r is notated as $deg(r) = |S^*(r)|$ for all $r \in V(G)$. The maximum and minimum degrees of the graph G are denoted by $\triangle(G)$ and $\delta(G)$, respectively. A set $S_d \subseteq V$ is called the Star Dominating Set (SRDS) if the star of r intersects the star of some vertex in S_d, for every vertex $r \in V - S_d$. The Star Domination Number (SRDN) is the minimum cardinality among all SRDSs of G. The function $f : V(G) \to [0, 1]$ is said to be the fractional dominating function if $\sum_{r\in Nbhd[y]} f(r) \geq 1$ for all $y \in V(G)$. The fractional domination number is the minimum weight of all FRDFs of G and is denoted by $\gamma_f(G)$.

The function $g : E(G) \to [0, 1]$ is known as the Fractional Star Dominating Function (FRSDF) of G, if it satisfies $\sum_{q\in S^*(r)} g(q) \geq 1$ for every $r \in V$, where $S^*(r)$ is the star of r. Here, $w(g) = \sum_{q\in E(G)} g(q)$ represents the weight. The Fractional Star Domination

Number (FRSDN) of a graph$G$ is the minimum weight of all FRSDFs of G and it is denoted by $\gamma_{gs}(G)$.

In[1] proposed many ideas that are related to split domination are proposed, and illustrated in some divisible dominating graphs. In[2,3] investigated some domination such as F-domination and total domination. The results of fractional graphs concerning the intuitionistic dominating functions are provided in[4]. In[5] introduced the concept of reinforcement numbers using half domination. The impacts of fractional domination and whole edge domination are provided when removing a vertex or removing or adding an edge from the graph, and some results related to the parameters are covered in[6] and[7]. The information on star partitioning is given in[8], and the bounds and algorithms of signed star domination numbers in some classes of graphs are introduced[9]. In[10] initiated the study of fractional eternal domination in graphs.

The star domination is merged with several other known dominating parameters. To the best of our understanding, the FRSDF has not yet been researched. In this paper, the fractional star dominating function of G is created by combining the fractional dominating function and star-dominating set. In this paper, we begin investigating the Fractional Star-Dominating Function (FRSDF), which is an extension of an edge fractional dominating function. Here, we add the additional condition that the sum of the function values of the edges incident at each vertex is at least one. In internet systems, rooted product graphs are used to link the internet from one system to another. The rooted product of a path P_r and a cycle $C_{r'}$ is a graph created by taking one copy of an r-vertex graph P_r and r copies of $C_{r'}$ and then merging the j-th vertex of P_r with all vertices of the j-th copy of $C_{r'}$. This graph is denoted by $P_r \odot C_{r'}$, refer[11].

In Section 2, we evaluate the exact values of FRSDN on some standard graphs and discuss the rooted product of paths and cycles. For any graph G without isolated vertices, we have $\gamma_{gs}(G) \geq \gamma_{f'}(G)$. In other words, the fractional star dominating function of a graph G implies the edge fractional dominating function of G, but the converse need not be true. Also, in Section 3, we show some of the bounds and sharpness.

2 Methodology

Some results about FRSDN

In this section, we obtain the exact value of the FRSDN for some common graphs, such as paths, cycles, a star graph, a complete graph, and a complete bipartite graph. The FRSDN of the rooted product of paths and cycles are also determined.

Observation 2.1[12] For any graph G, we have $\gamma_{f'}(G) = \gamma(L(G))$. Hence it follows that $\gamma_{f'}(C_n) = \gamma_f(C_n) = \frac{n}{3}$ and $\gamma_{f'}(P_n) = \gamma_f(P_{n-1}) = \left\lceil \frac{n-1}{3} \right\rceil$ ..

Theorem 2.2 For any path $P_{r'}$ with $r' \geq 2$, then $\gamma_{gs}(\mathscr{P}_{r'}) = \left\lceil \frac{r'}{2} \right\rceil$.

Proof Consult the graph $G = P_{r'}$, which has at least two vertices, as well as its vertex set $V(G) = \{r_i : \ 1 \leq i \leq r'\}$ and an edge set $E(G) = \{q_j : \ j = 1,\ 2,\ \ldots,\ r' - 1\}$. Here, we allot a value of one to each pendant edge of the graph G.

Case(i): When $r' = 2k+3$, $k \in N$. This case is divided into two sub-cases.

Sub case(i): A function g is defined as $g : E(G) \to [0, 1]$ by $g(q_j)$, where $1 \leq j \leq r' - 1$. In this case, we distribute the values $g(q_1) = g(q_{r'-1})= 1$ and $g(q_{2y+3}) = 1$ or $g(q_{2y+2}) = 1$ where, $y \in \mathbb{Z}$, $\left(0 \leq y < \left\lfloor \frac{r'-2}{2} \right\rfloor\right)$, and the rest of the edges are set to 0. Therefore, the weight of g is $\mathrm{w(g)} = \sum_{\mathrm{q} \in \varepsilon(\mathscr{G})} \mathrm{g(q)} = \sum \mathrm{g}\left(\mathrm{q_j}\right) = \left\lceil \frac{r'}{2} \right\rceil$.

Sub case(ii): Here, we provide the value $g(q_1) = g(q_{r'-1})= 1$, and we set the value of $\frac{1}{t}$ as $g(q_{2y+2})$ or $g(q_{2y+3})$ or $g(q_{\frac{r'-1}{2}})$, where $t \in Z > 0$, and we allocate the value of $1 - \frac{1}{t}$ as $g(q_{2y+3})$ or $g(q_{2y+2})$ or $g(q_{\frac{r'-1}{2}+1})$. In this case, we have the weight of g, which is $\mathrm{w(g)} = \sum_{\mathrm{q} \in \varepsilon(\mathscr{G})} \mathrm{g(q)} = \sum \mathrm{g}\left(\mathrm{q_j}\right) = \left\lceil \frac{r'}{2} \right\rceil$.

Case(ii): For $r' = 2k$, $k \in N$ and $r' = 3$.

For $r' = 3$, we have $g(q_1) = g(q_2) = 1$. Thus $\gamma_{\mathrm{gs}}(\mathscr{P}_3) = 2 = \left\lceil \frac{r'}{2} \right\rceil$. For $r' = 2k$, $k \in N$. We used $g(q_1) = g(q_{r'-1})= 1$ and $g(q_{2y+3})$ = 1, otherwise 0. Then the weight of g is $\mathrm{w(g)} = \sum_{\mathrm{q} \in \mathscr{E}(\mathscr{G})} \mathrm{g(q)} = \sum \mathrm{g}\left(\mathrm{q_j}\right) = \left\lceil \frac{r'}{2} \right\rceil$. As a result, in all of the preceding cases, $\gamma_{gs}(\mathscr{P}_{r'}) = \left\lceil \frac{r'}{2} \right\rceil$.

Corollary 2.3 If $\gamma_f(\mathscr{P}_{r'}) = \left\lceil \frac{r'}{3} \right\rceil$ and $\gamma_{gs}(\mathscr{P}_{r'}) = \left\lceil \frac{r'}{2} \right\rceil$, then $\gamma_f(P_{r'}) \leq \gamma_{gs}(P_{r'})$.

Lemma 2.4[13] For any graph without isolated vertices $\gamma_{ss'}(G) = \gamma_{s'}(G)$.

Theorem 2.5 Let G be any with no isolated vertices. Then $\gamma_{gs}(G) \geq \gamma_{f'}(G)$.

Proof Consider G to be a graph, and g to be a FRSDF of G such that $\gamma_{gs}(G) = \sum_{q \in E(G)} g(q)$. Here, $\sum_{q' \in Nbhd[q]} g(q') = S^*(x) + S^*(y) - g(q) \geq 1$ for every $q = xy \in E(G)$, where $S^*(x)$ and $S^*(y)$ are the stars of x and y. Therefore, g is a fractional edge dominating function of G. Hence $\gamma_{gs}(G) \geq \gamma_{f'}(G)$.

Theorem 2.6 For any cycle $C_{r'}$ with $r' \geq 3$, $\gamma_{gs}(C_{r'}) = \frac{r'}{2}$.

Proof Take $C_{r'}$ the graph that has at least three vertices and has $|V(G)| = r' = |E(G)|$.

Case(i): When r' is even.

Sub case(i): A function g is defined as $g : E(G) \to [0,\, 1]$ by $g(q_j)$, where $1 \leq j \leq r'$. In this case, for all $j = 1,\, 2,\, \ldots\, r'$ are assigned a value $g(q_j) = \frac{1}{2}$. Therefore, the weight of g is $w(g) = \sum_{q \in E(G)} g(q) = \sum g(q_j) = \frac{r'}{2}$.

Sub case(ii): In this case, we assign the value 1 to $g(q_{2k})$, where k is an integer which is $(1 \leq k \leq \frac{r'}{2})$; otherwise, we assign the value 0 to all remaining edges. Then the weight of g is $w(g) = \sum_{q \in E(G)} g(q) = \sum g(q_j) = \frac{r'}{2}$.

Case(ii): When r' is odd.

In this case, the only way to assign a value is to have $g(q_j) = \frac{1}{2}$ for all $j = 1,\, 2,\, \ldots\, r'$. Thus, the weight is $w(g) = \sum g(q_j) = \frac{r'}{2}$. Hence $\gamma_{gs}(C_{r'}) = \frac{r'}{2}$.

Corollary 2.7 If $\gamma_f(C_{r'}) = \frac{r'}{3}$ and $\gamma_{gs}(C_{r'}) = \frac{r'}{2}$, then $\gamma_{gs}(C_{r'}) > \gamma_f(C_{r'})$.

Theorem 2.8 For the star graph $K_{1,\, r'}$, then $\gamma_{gs}(K_{1,\, r'}) = r'$.

Proof Let $G = K_{1,\, r'}$ be a star graph with the vertex set $V(G) = \{p \cup p_i : i = 1, 2,\, \ldots\, , r'\}$, and the set of pendant edges $\{(pp_i) : i = 1, 2,\, \ldots,\, r'\}$.

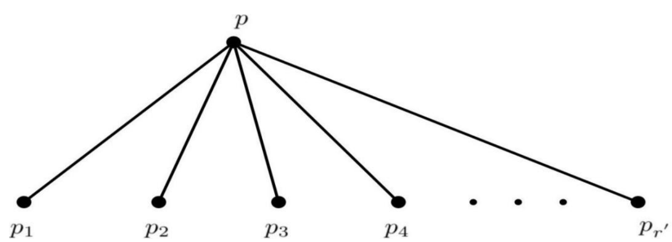

Fig 1. $K_{1,\, r'}$

We determine a function $g : E(G) \to [0,\, 1]$. If the function value of each pendant edge is 1, then the FRSDN of $K_{1,\, r'}$ is r'. Again, if the function value of some of the pendant edges is zero, then our assumption will be violated. As a result, all pendant edges can only have a value of 1. Hence, $\gamma_{gs}(G) = r'$.

Theorem 2.9 For $K_{r'}$ with $r' \geq 3$, then $\gamma_{gs}(K_{r'}) = \frac{q'}{r'-1}$.

Proof Consider $G = K_{r'}$ as a complete graph with $|V(G)| = r'$ and $|E(G)| = q' = \frac{(r')(r'-1)}{2}$, where q' is the cardinality of edges in G. We have a function $g : E(G) \to [0,\, 1]$ as follows; here, we assign the value $\frac{1}{r'-1}$ for all $q_i, (1 \leq i \leq \frac{(r')(r'-1)}{2})$. Thus, the weight of g is $w(g) = \sum_{q \in E(G)} g(q) = \frac{|E(G)|}{r'-1} = \frac{q'}{r'-1}$.

Corollary 2.10 If $K_{q',\, r'}$ is a complete bipartite graph, then $\gamma_{gs}(K_{q',\, r'}) = max\{q',\, r'\}$.

Theorem 2.11 If $P_r \odot C_{r'}$ has a rooted product with $r \geq 2$ and $r' \geq 3$, then $\gamma_{gs}(P_r \odot C_{r'}) = \frac{rr'}{2}$.

Proof Consider $G = P_r \odot C_{r'}$ as a rooted product of path and cycle graphs with $r \geq 2$ and $r' \geq 3$. A graph G has an edge set of $E(G) = \{(X \cup Y) = b_t \in E(G) : 1 \leq t \leq rr' + r - 1\}$ where $X = \{q_i' \in E(C_{r'}) : i = 1, 2, \ldots, rr'\}$ and $Y = \{q_j \in E(P_r) : j = 1, 2, \ldots, r-1\}$ and vertex set as $|V(G)| = rr'$. Here, we have three cases.

Case(i): When r is odd and r' is odd. In this case, a FRSDF $g : E(G) \to [0,\, 1]$ is created by $-$. Therefore, the weight of g is $w(g) = \sum_{b_t \in E(G)} g(b_t) = \sum (g(q_i') + g(q_j)) = \frac{rr'}{2}$.

Case(ii):

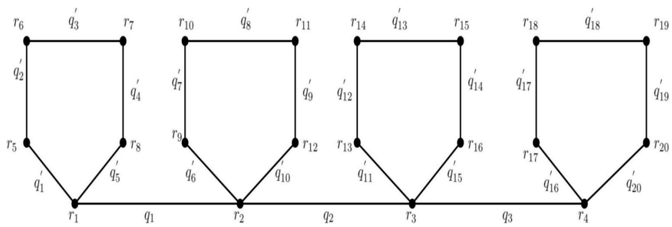

Fig 2. The fractional star domination of$P_4 \odot C_5$

Sub case(i): If r is odd and $r^{'}$ is even. Create a function $g : E(G) \to [0,\ 1]$ by the following two possibilities: $g\left(b_t\right) = \begin{cases} \frac{1}{u} & \text{for } q'_k \in X \\ 1-\frac{1}{u} & \text{for } q'_{k+1} \in X \\ 0 & \text{otherwise} \end{cases}$

where, $u \in Z > 0$ and $k \in Z$ as $(1 \le k \le \frac{rr^{'}}{2})$ (or) $-$. Thus, the weight of g is $w(g) = \sum_{b_t \in E(G)} g(b_t) = \sum (g(q_i^{'}) + g(q_j)) = \frac{rr^{'}}{2}$.

Sub case(ii): If r is even and $r^{'}$ is odd. Define a function $g : E(G) \to [0,\ 1]$ by the following two possibilities: $\mathrm{g}\left(\mathrm{b_t}\right) = \begin{cases} 1 \text{ for all } \mathrm{q_{2v-1}} \in \mathscr{Y} \text{ and } \mathrm{q'_{2k}} \in X \\ 0 & \text{otherwise} \end{cases}$

where, v and k are an integers, which is $(1 \le v \le \frac{r}{2})$ and $\left(1 \le k \le \left\lceil \frac{r'}{2} \right\rceil\right)$ (or) *otherwise*

$\mathrm{g(b_t)} = \begin{cases} \frac{1}{2} \text{ for all } \mathrm{q'_i} \in X \\ 0 & \text{otherwise} \end{cases}$. Then the weight of g is $w(g) = \sum_{b_t \in E(G)} g(b_t) = \sum (g(q_i^{'}) + g(q_j)) = \frac{rr^{'}}{2}$.

Sub sub case(i): If $r = 2$ and $r^{'}$is odd, then we assign the value $\frac{1}{u}$ for all $q_j \in Y$ and $q_i^{'}$ for some i and put $1 - \frac{1}{u}$ for $q_{i+1}^{'} \in X$, where u is any positive integer. Here, the weight of g is $w(g) = \sum_{b_t \in E(G)} g(b_t) = \frac{rr^{'}}{2}$.

Sub sub case(ii): If $r \ge 4$ and $r^{'}$is odd, then we assign the value $\frac{1}{2}$ for all $q_i^{'} \in X$ and 0 for all $q_j \in Y$. Thus, the weight of g is $w(g) = \sum_{b_t \in E(G)} g(b_t) = \frac{rr^{'}}{2}$.

Case(iii): When r is even and $r^{'}$ is even. We determine a function $g : E(G) \to [0,\ 1]$ by the following two possibilities:

$g\left(b_t\right) = \begin{cases} 1 \text{ for all } \mathrm{q'_{2k-1}} \in X \text{ and } \mathrm{q'_{2k}} \in X \\ 0 & \text{otherwise} \end{cases}$

where k as an integer, which is $(1 \le k \le \frac{rr^{'}}{2})$ (or) $-$. Therefore, the weight of g is $w(g) = \sum_{b_t \in E(G)} g(b_t) = \sum (g(q_i^{'}) + g(q_j)) = \frac{rr^{'}}{2}$. Hence, $\gamma_{gs}(P_r \odot C_{r^{'}}) = \frac{rr^{'}}{2}$.

3 Result and discussion

Bounds on $\gamma_{gs}(G)$

In this section, we present some bounds on the fractional star domination number of some general graphs and trees, as well as discuss the exactness of the bounds.

Theorem 3.1 If any graph of order c and size d has $\delta(G) \ge 1$, then $\gamma_{gs}(G) + d \ge c$.

Proof Here, G is a graph with $\delta(G) \ge 1$, where δ denotes the minimum degree. It has the order c and the size d.

To prove: $\gamma_{gs}(G) + d \ge c$. Let g be a FRSDF of G and $\sum_{q \in S^*(r)} g(q) \ge 1$ for each vertex $r \in V(G)$. Thus, $\gamma_{gs}(G) = w(g) = \sum_{q \in E(G)} g(q) \ge 1 \ge c - d$. This implies that $\gamma_{gs}(G) + d \ge c$. This bound is sharp for P_2.

Theorem 3.2 If any graph G has $\delta(G) \ge 1$, then $\gamma_{gs}(G) \ge \frac{r^{'}}{2}$.

Proof Assume G as a graph with $\delta(G) \ge 1$, where δ denotes the minimum degree, and its order is $r^{'}$. Let g be a FRSDF of G that is $\gamma_{gs}(G) = \sum_{q \in E(G)} g(q) = \frac{1}{2} \sum_{r \in V(G)} \sum_{q \in S^*(r)} g(q) \ge \frac{1}{2} \sum_{r \in V(G)} 1 = \frac{r}{2}$. Hence, $\gamma_{gs}(G) \ge \frac{r^{'}}{2}$. This bound is sharp for $P_{r^{'}}$, where $r^{'}$ is even, as well as for the graphs $C_{r^{'}}$, $K_{r^{'}}$, and $K_{c,d}$, where c = d, here c and d are the vertex cardinality of first and second partition of $K_{c,d}$.

Theorem 3.3 For any graph G with $\delta(G) \geq 1$, then $max\{\gamma_{f'}(G),\ \frac{r'}{2},\ r' - h_1\} \leq \gamma_{gs}(G) \leq (h_1 + h_2) - 1$, where h_1 and h_2 are the number of odd and even vertices of G, respectively.

Proof Consider G as a graph with $\delta(G) \geq 1$. Let r' be the order of G, and q' be the size of G, and h_1 and h_2 are the number of odd and even vertices of G. By theorems (2.5) and (3.2), we have $\gamma_{gs}(G) \geq \gamma_{f'}(G)$ and $\gamma_{gs}(G) \geq \frac{r'}{2}$.

Claim 1: $\gamma_{gs}(G) \geq r' - h_1$. Let g be a FRSDF of G and $\sum_{q \in S^*(r)} g(q) \geq 1$ for every vertex $r \in V(G)$. Take h_1 and h_2 as $h_1 = \{r_i :\ i\ is\ odd\}$ and $h_2 = \{r_j :\ j\ is\ even\}$. Here, $h_1 + h_2 = r'$. Enough, we must prove that $\gamma_{gs}(G) \geq h_2$. If $h_1 = h_2$ and $\triangle(G) \leq 2$, then $\gamma_{gs}(G) = w(g) = \sum_{q \in E(G)} g(q) \geq \frac{r'}{2} \geq \frac{h_1+h_2}{2} = h_2$. Thus, $\gamma_{gs}(G) = h_2$. Suppose that $h_1 = h_2$ and $\triangle(G) > 2$, we have $\gamma_{gs}(G) = w(g) = \sum_{q \in E(G)} g(q) \geq \frac{r'}{2} = \frac{h_1+h_2}{2} = h_2$. Thus, $\gamma_{gs}(G) = h_2$. If $h_1 \neq h_2$, then always $h_1 > h_2$. Here, $\gamma_{gs}(G) = w(g) = \sum_{q \in E(G)} g(q) \geq \frac{r'}{2} = \frac{h_1+h_2}{2} = \frac{1}{2}(\sum_{r_i \in h_1} 1 + \sum_{r_j \in 2} 1) = \frac{1}{2}(2h_2 + 1) > h_2$. Thus, $\gamma_{gs}(G) > h_2$. Therefore, $\gamma_{gs}(G) \geq h_2 = r' - h_1$.

Claim 2: $\gamma_{gs}(G) \leq (h_2 + h_1) - 1$. Enough to prove that $\gamma_{gs}(G) \leq q'$. Here, $\gamma_{gs}(G) = q'$ for a star graph; otherwise, $\gamma_{gs}(G) < q'$. Clearly, we have $\gamma_{gs}(G) \leq q' = r' - 1 = (h_1 + h_2) - 1$. Hence $max\{\gamma_{f'}(G),\ \frac{r'}{2},\ r' - h_1\} \leq \gamma_{gs}(G) \leq (h_1 + h_2) - 1$. For P_2, this bound becomes sharp.

Theorem 3.4 For any tree $\mathbf{T}$ with $\triangle(G) \geq 2$, we have $\gamma_{\mathrm{gs}}(\mathbf{T}) + \Delta(\mathbf{T}) \geq \max\left\{\gamma_f(\mathbf{T}) + \left\lceil \frac{r'}{q'} \right\rceil, \left\lceil \frac{|p(\varepsilon)|}{2} \right\rceil + 1\right\}$, where q' and r' are the size and order, and $|p(E)|$ is the cardinality of pendant edges.

Proof Think of $\mathbf{T}$ as a tree with $\triangle(T) \geq 2$ and g as a FRSDF of G.

Claim 1: $\gamma_{\mathrm{gs}}(\mathbf{T}) + \Delta(\mathbf{T}) \geq \gamma_f(\mathbf{T}) + \left\lceil \frac{r'}{q'} \right\rceil$.. Here, we have two cases.

Case(i): $\triangle(T) = 2$. For path graph, $\gamma_{gs}(T\ ) = \sum_{q \in E(T)} g(q) \geq \sum_{r \in V(T\ )} g(r) = \gamma_f(T\ )$. Thus $\gamma_{\mathrm{gs}}(\mathbf{T}) + \Delta(\mathbf{T}) \geq \gamma_f(\mathbf{T}) + \left\lceil \frac{r'}{q'} \right\rceil$..

Case(ii): $\triangle(T) > 2$. For any tree graph, $\gamma_{gs}(T\ ) = \sum_{q \in E(T)} g(q) \geq \sum_{r \in V(T\ )} g(r) = \gamma_f(T\ )$. Thus $\gamma_{\mathrm{gs}}(\mathbf{T}) + \Delta(\mathbf{T}) \geq \gamma_f(\mathbf{T}) + \left\lceil \frac{r'}{q'} \right\rceil$..

Claim 2: $\gamma_{\mathrm{gs}}(\mathbf{T}) + \Delta(\mathbf{T}) \geq \left\lceil \frac{|\mathfrak{p}(\mathscr{E})|}{2} \right\rceil + 1$.

Enough to prove that, $\gamma_{gs}(T\ ) \geq |p(E)|$. For the star graph, $\gamma_{gs}(T\ ) = |p(E)|$ and path graph $\gamma_{gs}(T\ ) \geq |p(E)|$. Thus, for any tree, $\gamma_{gs}(T\ ) = \sum_{q \in E(T)} g(q) \geq |p(E)|$. We have $\gamma_{\mathrm{gs}}(\mathbf{T}) + \Delta(\mathbf{T}) \geq |\mathfrak{p}(\mathscr{E})| \geq \left\lceil \frac{|\mathfrak{p}(\mathscr{E})|}{2} \right\rceil + 1$ for $\triangle \geq 2$. Finally, we have $\gamma_{\mathrm{gs}}(\mathbf{T}) + \Delta(\mathbf{T}) \geq \max\left\{\gamma_f(\mathbf{T}) + \left\lceil \frac{r'}{q'} \right\rceil, \left\lceil \frac{|p(\mathscr{E})|}{2} \right\rceil + 1\right\}$ from the previous two inequalities. This completes the proof. Also, this bound is sharp for path P_4.

Fig 3. Sharpness onP_4=P_4

Figure 3 provides the equality of the theorem (3.4) bound.

Proposition 3.5 [10] For any $n \in N$:

1. $\gamma_f^{\infty}(K_n) = 1$.
2. $\gamma_f^{\infty}(P_n) = \alpha(P_n) = \left\lceil \frac{n}{2} \right\rceil$.
3. $\gamma_f^{\infty}(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$ if n ≥ 3.

Observation 3.6 [7]

1.For the path graph P_n with n ≥ 3, $\gamma_{\mathrm{whe}}\ (P_n) = \begin{cases} 1 \text{ if } 3 \leq n \leq 4 \\ P_n \text{ has no WDES, Otherwise} \end{cases}$.

2.For the cycle graph C_n with n ≥ 3, $\gamma_{\mathrm{whe}}\ (C_n) = \begin{cases} 1 \text{ if } n = 3 \\ 2 \text{ if } n = 4 \\ C_n \text{ has no } W\ DES, \text{ Otherwise} \end{cases}$.

3.For the complete graph K_n with n ≥ 3, $\gamma_{\mathrm{whe}}\ (K_n) = \begin{cases} 1 \text{ if } n = 3 \\ 2 \text{ if } n = 4 \\ K_n \text{ has no WDES, Otherwise} \end{cases}$.

4.For a Wheel graph W_n with n ≥ 3, $\gamma_{\mathrm{whe}}\ (W_n) = \begin{cases} 2 \text{ if } n = 3 \\ W_n \text{ has no } W\ DES, \text{ Otherwise} \end{cases}$.

5.For star graph S_n with n $\geq 3, \gamma_{\text{whe}}\ (S_n) = 1$.

Note 3.7:

For the previous theorems (2.2), (2.6), (2.8) and (2.9), proposition (3.5) and the observation (3.6), we conclude the following comparisons here $r' = n$:

(i) If $\gamma_{gs}(K_{r'}) = \frac{q'}{r'-1}$ and $\gamma_f^{\infty}(K_{r'}) = 1$, then $\gamma_{gs}(K_{r'}) > \gamma_f^{\infty}(K_{r'})$.

(ii) If$\gamma_{\text{gs}}(\mathscr{P}_{r'}) = \left\lceil \frac{r'}{2} \right\rceil$and $\gamma_f^{\infty}(P_{r'}) = \alpha(P_{r'}) = \left\lceil \frac{r'}{2} \right\rceil$, then $\gamma_{gs}(P_{r'}) = \gamma_f^{\infty}(P_{r'}) = \alpha(P_{r'})$.

(iii) If $\gamma_{gs}(\mathscr{C}_{r'}) = \frac{r'}{2}$ and $\gamma_f^{\infty}(C_{r'}) = \gamma(C_{r'}) = \left\lceil \frac{r'}{3} \right\rceil$, then $\gamma_{\text{gs}}(\mathscr{C}_{r'}) \geq \gamma_f^{\infty}(C_{r'})$.

(iv) If $\gamma_{gS}(\mathscr{P}_{r'}) = \left\lceil \frac{r'}{2} \right\rceil$ and $\gamma_{whe}(C_{r'}) = \begin{cases} 1 \text{ if } n = 3 \\ 2 \text{ if } n = 4 \\ C_{r'} \text{ has no WDES, Otherwise} \end{cases}$, then $\gamma_{gs}(P_{r'}) > \gamma_{whe}(P_{r'})$ with $r' = 3, 4$.

(v) If $\gamma_{gs}(C_{r'}) = \frac{r'}{2}$ and $\gamma_{whe}(C_{r'}) = \begin{cases} 1 \text{ if } n = 3 \\ 2 \text{ if } n = 4 \\ C_{r'} \text{ has no WDES, Otherwise} \end{cases}$, then $\gamma_{gs}(C_{r'}) > \gamma_{whe}(C_{r'})$ with $r' = 3$ and $\gamma_{gs}(C_{r'}) = \gamma_{whe}(C_{r'})$ with $r' = 4$.

(vi) If $\gamma_{gs}(K_{r'}) = \frac{q'}{r'-1}$ and $\gamma_{whe}(K_{r'}) = \begin{cases} 1 \text{ if } n = 3 \\ 2 \text{ if } n = 4 \\ K_n \text{ has no WDES, Otherwise} \end{cases}$, then $\gamma_{gs}(K_{r'}) > \gamma_{whe}(K_{r'})$ with $r' = 3$ and $\gamma_{gs}(K_{r'}) = \gamma_{whe}(K_{r'})$ with $r' = 4$.

(vii) If $\gamma_{\text{gs}}\left(\mathscr{K}_{1,r'}\right) = r'$ and $\gamma_{whe}(S_n) = 1$, then $\gamma_{\text{gs}}\left(\mathscr{K}_{1,r'}\right) > \gamma_{whe}(S_n)$.

4 Conclusion

This study presents a new parameter called fractional star domination, which combines the fractional dominating function and star domination. The exact value of the FRSDN of some known graphs, such as paths, cycles, and trees have been calculated. Here, the accurate value of the FRSDN of the rooted product of paths and cycles are also obtained. In addition, the bounds for fractional star domination on some general graphs and trees were determined, as well as the sharpness of the bound. The inequality of the graphs for any simple graph G are $\gamma_{gs}(G) \geq \gamma_{f'}(G)$. In the future, the changing and unchanging fractional star domination when a vertex is erased or an edge is eliminated or an edge is inserted to G may be studied.

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