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RESEARCH ARTICLE

Isomorphism on Complex Fuzzy Graph

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Abstract

Objective: To investigate isomorphism between two complex fuzzy graphs and prove it is an equivalence relation. The major objective of this research paper is to elucidate weak and strong isomorphism, and the study also endeavours to look at the complement of complex fuzzy graphs. **Methods:** Isomorphism is examined by comparing the membership values of nodes and arcs (both amplitude and phase). The same technique also proves further validation of the equivalence relation, which is also proven by the same technique. This criterion helps us to identify and formalise the isomorphic relationship between complex fuzzy graphs. **Findings:** This study demonstrates that isomorphism in complex fuzzy graphs allows for consideration of graph structures' differences and similarities. It provides for a better understanding of significant connections between complex fuzzy graphs. **Novelty:** By offering an elaborate perception of the idea of isomorphism in complex fuzzy graphs, this work advances the discipline. It seeks a better understanding of the complexities of equivalency determination for complicated structures, as well as the application of isomorphism theory to complex fuzzy graphs.

Keywords: Complex fuzzy graph; Homomorphism; Isomorphism; Partial order relation; Self complementary

1 Introduction

Each fuzzy graph element's membership value falls between 0 and 1. It will be extended to the unit circle of the complex plane in the complex fuzzy graph (CFG). This study explores the concept of membership values in polar form (both amplitude and phase value). Naveed Yaqoob et al[1], described the homomorphism of Complex Intuitionistic Fuzzy Graphs (CIFG). In CIFG, they have introduced certain operations, such as union, join, and composition. Moreover, Abida Anwar[2], defined the strong CIFG and discussed the product of two CIFGs. In CFG, Shoaib et al[3], discovered the new maximal product and symmetric difference operations in CFG. Shoaib et al[4], elaborated on the basic operations in complex spherical fuzzy graphs (CSFG) along with applications, which gives us the ability to deal with problems in three directions. E.M. Ahamed Buttet et al[5], implemented the dombi operators in complex fuzzy sets to extend the graph to a complex dombi fuzzy graph (CDFG). They discussed the concepts of homomorphism, weak isomorphism, and co-weak isomorphism between two CDFGs and established the application of CDFG in decision-making problems. Shoaib et al[6], elaborated on the fundamental operations in complex picture fuzzy graphs (CPFG), such as union, complement, and cartesian product. They discussed the application of CPFGs to decision-making problems. N. Azhagendran and A. Mohamed Ismayil[7], proved some basic theorems based on operations on CFGs, with examples. The characteristics and operations of CFGS are discussed by Veeramani and Suresh[8]. Almutairi et al[9], discussed the homomorphic and isomorphic relationships of complex anti-fuzzy subgroups over group homomorphism. The homomorphism and isomorphism of t-intuitionistic fuzzy graphs was discussed by Asima Razzaque et al[10]. This study identifies isomorphisms between complex fuzzy graphs, co-strong isomorphisms, and self-complementary of complex fuzzy graphs.

2 Methodology

Definition 2.1.

Let U be a universe of discourse and $Z \subseteq U$. A complex fuzzy set (CFS) σ_c is defined by $\sigma_c = \left\{ z/r(z)\, e^{i\theta(z)} : z \in Z \right\}$ where $r(z)$ is an amplitude and $\theta(z)$ is a phase term of z, $i = \sqrt{-1}$, $0 \le r(z) \le 1$ *and* $0 \le \theta(z) \le 2\pi$.

Example 2.2. An example for complex fuzzy set σ_c is given below.

$\sigma_c = \{z_1 \,/\, 0.2\, e^{i\pi}, z_2 \,/\, 0.8\, e^{i0.5\pi}, z_3 \,/\, 0.3\, e^{i2\pi}, z_4 \,/\, 0.25\}$

Definition 2.3.

A complex fuzzy graph (CFG) $G_c = (\sigma_c, \mu_c)$ defined on a graph $G = (V,E)$ is a pair of complex functions $\sigma_c : V \to r(z)\, e^{i\theta(z)}$, $\mu_c : E \subseteq V \times V \to R(e)\, e^{i\phi(e)}$ such that

$\mu_c(z_1,z_2) = R(e)\; e^{i\phi(e)}$, where $R(e) \le min\{r(z_1), r(z_2)\}$ and $\phi(e) \le min\{\theta(z_1), \theta(z_2)\}$ for all $z_1, z_2 \in V$ and $0 \le r(z_1), r(z_2) \le 1$ *and* $0 \le \theta(z_1), \theta(z_2) \le 2\pi$.

Example 2.4. $G_c = (\sigma_c, \mu_c)$ is a CFG, where $\sigma_c = \{z_1 \,/\, 0.4\, e^{i\pi}, z_2 \,/ 0.6\, e^{i0.5\pi}, z_3 \,/\, 0.8\, e^{i\pi}\}$, $\mu_c = ((z_1, z_2)/0.3\, e^{i0.5\pi}, (z_2, z_3)/\, 0.4\, e^{i0.2\pi}, (z_1,z_3) \,/\, 0.4\}$.

Definition 2.5.

The order p and size q of a CFG, $G_c = (\sigma_c, \mu_c)$ on $G = (V,E)$ are defined by

$p = \sum_{z\in V} r(z)\,.\;\; e^{i\,\sum_{z\in V}\theta(z)}$; $q = \sum_{e\in E} R(e)\,.\;\; e^{i\,\sum_{e\in E}\phi(e)}$

Note: The CFG of order p and size q is denoted by (p,q) CFG.

Definition 2.6.

The degree of a vertex z_i, in a CFG, $G_c = (\sigma_c, \mu_c)$ on $G = (V,E)$ is defined by $d(z_i) = \sum_{e=(z_i,z_j)\in\mu_c} R(e)\,.\;\; e^{i.\;\sum_{e=(z_i,z_j)\in\mu_c}\phi(e)}$ such that $\mu_c(z_i,z_j) = R(e)\, e^{i\phi(e)}$, for all $z_j \in \sigma_c$.

Definition 2.7.

A CFG $\overline{G_c} = (\bar{\sigma}_c, \overline{\mu_c})$ is said to be a complement of CFG G_c if

i) $\bar{\sigma}_c(z) = \sigma_c(z)$ and

ii) $\bar{\mu}_c(z_1,z_2) = \bar{R}(e) e^{i\bar{\phi}(e)}$, where $\bar{R}(e) = min\{r(z_1), r(z_2)\} - R(e)$ and $\bar{\phi}(e) = min\{\theta(z_1), \theta(z_2)\} - \phi(e)$, for all $z_1, z_2 \in V$.

3 Results and Discussion

Definition 3.1.

Let $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$and $G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$ are two complex fuzzy graphs defined on $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. Then the homomorphism $h : G_{c_1} \to G_{c_2}$is a mapping $h : V_1 \to V_2$

a)$r_1(z_1).e^{i\theta_1(z_1)} \le r_2(h(z_1)).e^{i\theta_2(h(z_1))}$, for all z_1 in V_1

b)$R_1(e).e^{i\varphi_1(e)} \le R_2(h(e)).e^{i\varphi_2(h(e))}$, for all $e = (z_1, z_2) \in E_1$.

A One-one, onto homomorphism satisfying the property

c) $r_1(z_1) \cdot e^{i\theta_1(z_1)} = r_2(h(z_1)) \cdot e^{i\theta_2(h(z_1))}$ for all z_1 in V_1 then it is said to be weak isomorphism.

A one-one, onto homomorphism satisfying the property

d) $R_1(e) \cdot e^{i\varphi_1(e)} = R_2(h(e)) \cdot e^{i\varphi_2(h(e))}$, for all $e = (z_1, z_2) \in E_1$.

Then it is said to be strong co-isomorphism.

A bijective mapping satisfying the conditions c) and d) then it is said to be isomorphism.

Example 3.2. Consider the following two complex fuzzy graphs $G_{c_1} = (\sigma_{c_1}, \mu_{c_1})$and $G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$ where $\sigma_{c_1} = \{z_1/0.4.e^{i0.7\pi}, z_2/0.6.e^{i0.4\pi}\}$, $\mu_{c_1} = \{(z_1,z_2)/0.4.e^{i0.4\pi}$ and $\sigma_{c_2} = z_3/0.6.e^{i0.4\pi}, z_4/0.4.e^{i0.7\pi}$, $\mu_{c_1} = \{(z_3,z_4)/0.4.e^{i0.2\pi}$. Clearly it is weak isomorphism $h : V_1 \to V_2$ with $h(z_1) = z_4$ *and* $h(z_2) = z_3$.

Theorem 3.3.

If G_{c_1} *and* G_{c_2} are isomorphic complex fuzzy graphs then the order and size of G_{c_1} is same as that of G_{c_2}.

Proof. Let G_{c_1} and G_{c_2} are isomorphic complex fuzzy graphs and let the order and size of G_{c_1} are p_1 and q_1 respectively and the order and size of G_{c_2} are p_2 *and* q_2 *respectively.* If $h : G_{c_1} \to G_{c_2}$ is an isomorphism then by definition 3.1

then $r_1(z_1).e^{i\theta_1(z_1)} = r_2(h(z_1)).e^{i\theta_2(h(z_1))}$, *for all* z_1 *in* V_1 and $R(e).e^{i\varphi_1(e)} = R_2(h(e)).e^{i\varphi_1(h(e))}$, *for all* $e = (z_1, z_2) \in E_1$.

i) order of $G_{c_1} = p_1 = \sum_{z_1\in V_1} r_1(z_1)\cdot e^{\sum_{z_1\in V_1}\theta_1(z_1)} = \sum_{z_1\in V_1} r_2(h(z_1))\cdot e^{\sum z_1\in V_1\theta_2(h(z_1))}$

$= \sum_{z_1\in V_1} r_2(z_1')\cdot e^{\sum_{z_1\in V_1}\theta_2(z_1')} = p_2$

ii) size of $G_{c_1} = q_1 = \sum_{e=(z_1,z_2)\in E_1} R_1(e)\cdot e^{i\sum_{e=(z_1,z_2)\in E_1}\cdot\varphi_1(e)}$

$= \sum_{e=(z_1,z_2)\in E_1} R_1(z_1,z_2)\cdot e^{i\sum_{e=(z_1,z_2)\in E_1}\cdot\varphi_1(z_1,z_2)}$

$= \sum_{e=(z_1,z_2)\in E_1} R_2(h(z_1), h(z_2))\cdot e^{i\varphi_2(h(z_1),h(z_2))}$

$= \sum_{e=(z_1,z_2)\in E_1} R_2(z_1',z_2')\cdot e^{i\varphi_2(z_1',z_2')}$

$= \sum_{e=(z_1,z_2)\in E_1}. R_2(h(e))\cdot e^{i\sum_{e=(z_1,z_2)\in E_1}\cdot\varphi_2(h(e))}$

Hence isomorphic complex fuzzy graphs whose order and size are same.

Corollary 3.4. Converse of the theorem 3.1 need not be true.

Example 3.5. Consider the G_{c_1} and G_{c_2} are two complex fuzzy graphs defined by

$\sigma_{c_1} = \{z_1/0.8.e^{i1.2\pi}, z_2/0.6.e^{i1.4\pi}, z_3/0.4.e^{i0.6\pi}, z_4/0.6.e^{i0.2\pi}\}, \mu_{c_1} = \{(z_1,z_3)/0.2.e^{i0.4\pi}, (z_3,z_4)/0.4.e^{i0.2\pi}, (z_2,z_4)/0.2.e^{i0.2\pi}\}$ and

$\sigma_{c_2} = \{z_5/0.6.e^{i0.2\pi}, z_6/0.8.e^{i1.2\pi}, z_7/0.6.e^{i1.4\pi}, z_8/0.4.e^{i0.6\pi}\}, \mu_{c_2} = \{(z_5,z_8)/0.3.e^{i0.2\pi}, (z_6,z_8)/0.4.e^{i0.2\pi}, (z_7,z_8)/0.1.e^{i0.4\pi}\}$

in the above graph G_{c_1}, the order p_1=2.4 $e^{i1.4\pi}$ and size $q_1 = 0.8\ e^{i0.8\pi}$. Similarly, in G_{c_2} the order $p_2 = 2.4\ e^{i1.4\pi}$ and size $q_2 = 0.8\ e^{i0.8\pi}$.

The order and size of the above two complex fuzzy graphs are same. But clearly G_{c_1} is not isomorphic to G_{c_2}. Because the degree of any vertex in G_{c_1} is not same as the degree of any vertex in G_{c_2}.

Remarks 3.6. The order of two CFG are same in weak isomorphic CFG but converse need not be true.

Example 3.7. Consider G_{c_1} and G_{c_2} are two complex fuzzy graphs defined by

$\sigma_{c_1} = \{z_1/0.8\ e^{i0.4\pi}, z_2/0.6.e^{i\pi}, z_3/0.7\ e^{i0.2\pi}, z_4/0.5\ e^{i0.5\pi}\}$,

$\mu_{c_1} = \{(z_1,z_2)/0.5\ e^{i0.4\pi}, (z_2,z_3)/0.6\ e^{i0.2\pi}, (z_3,z_4)/0.5\ e^{i0.2\pi}, (z_1,z_4)/0.4\ e^{i0.\pi}\}$

and $\sigma_{c_2} = \{z_5/1\ e^{i0.2\pi}, z_6/0.4\ e^{i0.8\pi}, z_7/0.8\ e^{i1.2\pi}, z_8/0.4\ e^{i0.2\pi}\}$,

$\mu_{c_2} = \{(z_5,z_6)/0.4\ e^{i0.2\pi}, (z_6,z_7)/0.3\ e^{i0.4\pi}, (z_7,z_8)/0.4\ e^{i0.1\pi}, (z_5,z_8)/0.4\ e^{i0.2\pi}\}$

The order of above two complex fuzzy graphs is same (i.e)p_1=2.6 $e^{i2.2\pi} = p_2$. But they are not weak isomorphic.

Theorem 3.8.

An isomorphism between complex fuzzy graphs is an equivalence relation.

Proof. Let $G_{c_1} = (\sigma_{c_1}, \mu_{c_1}), G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$ *and* $G_{c_2} = (\sigma_{c_3}, \mu_{c_3})$ are complex fuzzy graphs defined on $G_1 = (V_1, E_1), G_2 = (V_2, E_2), G_3 = (V_3, E_3)$ respectively. Here $\sigma_{c_1} : V_1 \to r_1(z).e^{i\theta_1(z)}, \mu_{c_1} : E_1 \subseteq V_1\times V_1 \to R_1(e).\ e^{i\varphi_1(e)}$, $\sigma_{c_2} : V_2 \to r_2(z).e^{i\theta_2(z)}, \mu_{c_1} : E_2 \subseteq V_2\times V_2 \to R_2(e).\ e^{i\varphi_2(e)}$, $\sigma_{c_3} : V_3 \to r_3(z).e^{i\theta_3(z)}, \mu_{c_1} : E_3 \subseteq V_3\times V_3 \to R_3(e).\ e^{i\varphi_3(e)}$.

We prove that it is reflexive, symmetric and transitive.

i) Reflexive: Define a map $h : V_1 \to V_1$ is an isomorphism from G_{c_1} to G_{c_1} such that

$h(z) = z, \forall z\ \in V_1$.

(i.e) $r_1(z).e^{i\theta_1(z)} = r_1(h(z)).e^{i\theta_1(h(z))}\forall\ z \in V_1$ and $R_1(e).e^{i\varphi_1(e)} = R_1(h(e)).e^{i\varphi_1(h(e))}$, for all $e = (z_1, z_2) \in E_1$. The condition is satisfied by the bijective function $h : V_1 \to V_1$. It is reflexive.

ii) Symmetric: Define a map $h_1 : V_1 \to V_2$ is an isomorphism from G_{c_1} to G_{c_2}.

As h is 1-1, onto map. Then

$r_1(z_1)\cdot e^{i\theta_1(z_1)} = r_2(h_1(z_1))\cdot e^{i\theta_2(h_1(z_1))}$

$= r_2\left(z_1'\right)e^{i\theta_2\left(z_1'\right)}, \forall z_1 \in V_1 \qquad (A)$

and

$R_1(e)\cdot e^{i\varphi_1(e)} = R_2(h_1(e))\cdot e^{i\varphi_2(h_1(e))}$

$= R_2(h_1(z_1), h_1(z_2))\cdot e^{i\varphi_2(h_1(z_1),h_1(z_2))}$

$= R_2(z_1', z_2')\cdot e^{i\varphi_2(z_1',z_2')}$

$= R_2\left(e'\right)\cdot e^{i\varphi_2\left(e'\right)}, \forall e = (z_1, z_2) \in E_1 \qquad (B)$

Since h is 1-1, by (A) $r_2[h_1^{-1}(z_1^{'})] = z_1, \forall\, z_1^{'} \in V_2$

Using (B)$R_1[h_1^{-1}(z_1^{'}), h_2^{-1}(z_2^{'})].e^{i\varphi_1(h_1^{-1}(z_1^{'}),h_2^{-1}(z_2^{'}))} = R_2(e^{'}).\ e^{i\varphi_2(e^{'})}, \forall e = (z_1, z_2) \in E_1$

Hence $h_1^{-1}$is 1-1and onto map$h_1^{-1} : V_2 \to V_1$ is an isomorphism (i.e)$G_{c_2} \cong G_{c_1}$

iii) Transitive: Define a map $h_1 : V_1 \to V_2$and $h_2 : V_2 \to V_3$ then $h_2 \circ h_1 : V_1 \to V_3$ is 1-1 and onto map.$r_1((h_2 \circ h_1)(z_1)).e^{i\theta_1((h_2\circ h_1)(z_1))} = r_3(z_1^{''}).e^{i\theta_3(z_1^{''})}, \forall\, z \in V_1$

And $h_1 : V_1 \to V_2, h_2 : V_2 \to V_3$ defined by $r_1(z_1).e^{i\theta_1(z_1)} = r_2(h_1(z_1)).e^{i\theta_2(h_1(z_1))} = r_2(z_1^{'})\ e^{i\theta_2(z_1^{'})}, \forall\, z_1 \in V_1$, $r_2(z_1^{'}).e^{i\theta_2(z_1^{'})} = r_3(h_2(z_1^{'})).e^{i\theta_3\left(h_2(z_1^{'})\right)} = r_2(z_1^{''})\ e^{i\theta_3(z_1^{''})}, \forall\, z_1 \in V_1$

Now, $R_1(e).e^{i\varphi_1(e)} = R_1(z_1, z_2).e^{i\varphi_2(z_1,z_2)} = R_2(h_1(e)).e^{i\varphi_2(h_1(e))}$=$R_2(h_1(z_1), h_1(z_2)).e^{i\varphi_2(h_1(z_1),h_1(z_2))}$

=$R_2(e^{'}).\ e^{i\varphi_2(e^{'})}, \forall e = (z_1, z_2) \in E_1$.

Similarly, we can prove $r_2(z_1^{'}).e^{i\theta_2\left(z_1^{'}\right)} = r_3(h_2(z_1^{'})).e^{i\theta_3\left(h_2(z_1^{'})\right)} = r_3(z_1^{''})\ e^{i\theta_3(z_1^{''})}, \forall\, z_1^{'} \in V_2$

Now, $R_2(e').e^{i\varphi_1(e')} = R_2(z_1^{'}, z_2^{'}).e^{i\varphi_2(z_1^{'},z_2^{'})} = R_3(h_2(e')).e^{i\varphi_2(h_2(e'))}$

=$R_3(h_2(z_1^{'}), h_2(z_2^{'})).e^{i\varphi_2((h_2(z_1^{'}),h_2(z_2^{'}))}$

=$R_3(e'').\ e^{i\varphi_3(e'')}, \forall e' = (z_1', z_2') \in E_2$.

From all the above equations, we can write

$r_1(z_1) \cdot e^{i\theta_1(z_1)} = r_2(h_1(z_1)) \cdot e^{i\theta_2(h_1(z_1))}, \forall z_1 \in V_1$

$= r_2(z_1^{'})\ e^{i\theta_2(z_1^{'})}$

$= r_3(h_2(z_1^{'})).e^{i\theta_3\left(h_2(z_1^{'})\right)}$

$= r_3(z_1^{''})\ e^{i\theta_3(z_1^{''})}, \forall\, z_1 \in V_1$

And

$R_1(e).e^{i\varphi_1(e)} = R_1(z_1, z_2).e^{i\varphi_2(z_1,z_2)}$

$= R_2(h_1(e)).e^{i\varphi_2(h_1(e))}$

=$R_2(h_1(z_1), h_1(z_2)).e^{i\varphi_2(h_1(z_1),h_1(z_2))}$

=$R_2(z_1^{'}, z_2^{'}).e^{i\varphi_2(z_1^{'},z_2^{'})}$

=$R_2(e^{'}).\ e^{i\varphi_2(e^{'})}, \forall e = (z_1, z_2) \in E_1$

=$R_3(h_2(z_1^{'}), h_2(z_2^{'})).e^{i\varphi_2(h_2(z_1^{'}),h_2(z_2^{'}))}$

=$R_3(z_1^{''}, z_2^{''}).e^{i\varphi_3(z_1^{''},z_2^{''})}$

=$R_3(e'').\ e^{i\varphi_3(e'')}, \forall e = (z_1, z_2) \in E_1$

Therefore $h_2 \circ h_1$ is an isomorphism from $G_{c_1} to\ G_{c_3}$. Hence it is an equivalence relation.

Theorem 3.9.

Weak isomorphism between complex fuzzy graphs is a partial order relation.

Proof. Let $G_{c_1} = (\sigma_{c_1}, \mu_{c_1}), G_{c_2} = (\sigma_{c_2}, \mu_{c_2})$ *and* $G_{c_2} = \left(\sigma_{c_3}, \mu_{c_3}\right)$ are complex fuzzy graphs defined on $G_1 = (V_1, E_1), G_2 = (V_2, E_2), G_3 = (V_3, E_3)$ respectively. Here $\sigma_{c_1} : V_1 \to r_1(z).e^{i\theta_1(z)}, \mu_{c_1} : E_1 \subseteq V_1 \times V_1 \to R_1(e).\ e^{i\varphi_1(e)}$,$\sigma_{c_2} : V_2 \to r_2(z).e^{i\theta_2(z)}$, $\mu_{c_1} : E_2 \subseteq V_2 \times V_2 \to R_2(e).\ e^{i\varphi_2(e)}$,$\sigma_{c_3} : V_3 \to r_3(z).e^{i\theta_3(z)}, \mu_{c_1} : E_3 \subseteq V_3 \times V_3 \to R_3(e).\ e^{i\varphi_3(e)}$.

We prove that it is reflexive, anti-symmetric and transitive.

i) Reflexive: Define a map $h : V_1 \to V_1$ such that $h(z) = z, \forall z\ \in V_1$

(i.e) $r_1(z).e^{i\theta_1(z)} = r_1(h(z)).e^{i\theta_1(h(z))} \forall\, z \in V_1$

and

$R_1(e).e^{i\varphi_1(e)} \leq R_1(h(e)).e^{i\varphi_1(h(e))}$, for all $e = (z_1, z_2) \in E_1$.

The condition is satisfied by the bijective function $h : V_1 \to V_1$. It is reflexive.

ii) Anti-symmetric

Define a map $h_1 : V_1 \to V_2$ and $h_2 : V_2 \to V_1$(i.e) h_1 is a weak isomorphism from $G_{c_1} to\ G_{c_2}$ and h_2 is a weak isomorphism from $G_{c_2} to\ G_{c_1}$.

We must prove that $G_{c_1} and\ G_{c_2}$ are identical

Then

$r_1(z_1).e^{i\theta_1(z_1)} = r_2(h_1(z_1)).e^{i\theta_2(h_1(z_1))}$

=$r_2(z_1^{'})\ e^{i\theta_2(z_1^{'})}, \forall\, z_1 \in V_1$

And

$R_1(e).e^{i\varphi_1(e)} \le R_2(h_1(e)).e^{i\varphi_2(h_1(e))}$
$=R_2(h_1(z_1),h_1(z_2)).e^{i\varphi_2(h_1(z_1),h_1(z_2))}$
$=R_2(z_1^{'},z_2^{'}).e^{i\varphi_2(z_1^{'},z_2^{'})}$
$=R_2(e^{'}).\ e^{i\varphi_2(e^{'})}, \forall e=(z_1,z_2)\in E_1$
Similarly,
$r_2(z_1^{'}).e^{i\theta_1\left(z_1^{'}\right)} = r_2(h_2(z_1^{'})).e^{i\theta_2\left(h_2(z_1^{'})\right)}$
$= r_1(z_1).e^{i\theta_1(z_1)}, \forall\, z_1^{'} \in V_2$
$R_2\left(e'\right)\cdot e^{i\varphi_1\left(e'\right)} \le R_2\left(z_1',z_2'\right)\cdot e^{i\varphi_2\left(z_1',z_2'\right)}$
$= R_2(h_2(e')).e^{i\varphi_2(h_2(e'))}$
$=R_3(h_2(z_1^{'}),h_2(z_2^{'})).e^{i\varphi_2((h_2(z_1^{'}),h_2(z_2^{'}))}$
$=R_1(z_1,z_2).e^{i\varphi_2(z_1,z_2)}$
$=R_1(e).\ e^{i\varphi_3(e)}, \forall e'=(z_1',z_2')\in E_2.$
Therefore $G_{c_1} and G_{c_2}$ are identical.
Hence it is anti-symmetric.
iii) Transitive: Define a map $h_1: V_1 \to V_2$and $h_2: V_2 \to V_3$ then $h_2 \circ h_1 : V_1 \to V_3$ is 1-1and onto map.
$r_1((h_2\circ h_1)(z_1)).e^{i\theta_1((h_2\circ h_1)(z_1))} = r_3(z_1^{''}).e^{i\theta_3(z_1^{''})}, \forall\, z\in V_1$
And $h_1: V_1 \to V_2$, $h_2: V_2 \to V_3$ defined by $r_1(z_1).e^{i\theta_1(z_1)} \;\; = r_2(h_1(z_1)).e^{i\theta_2(h_1(z_1))}$
$= r_2(z_1^{'})\ e^{i\theta_2(z_1^{'})}, \forall\, z_1 \in V_1$
$r_2(z_1^{'}).e^{i\theta_2(z_1^{'})} = r_3(h_2(z_1^{'})).e^{i\theta_3\left(h_2(z_1^{'})\right)}$
$= r_2(z_1^{''})\ e^{i\theta_3(z_1^{''})}, \forall\, z_1 \in V_1$
Now, $R_1\left(e\right).e^{i\varphi_1(e)} \le R_1(z_1,z_2).e^{i\varphi_2(z_1,z_2)}$
$= R_2(h_1(e)).e^{i\varphi_2(h_1(e))}$
$=R_2(h_1(z_1),h_1(z_2)).e^{i\varphi_2(h_1(z_1),h_1(z_2))}$
$$= R_2\left(e^{'}\right)\cdot e^{i\varphi_2\left(e^{'}\right)}, \quad \forall e=(z_1,z_2)\in E_1 \qquad (C)$$
Similarly, we can prove $r_2(z_1^{'}).e^{i\theta_2\left(z_1^{'}\right)} \;\; = r_3(h_2(z_1^{'})).e^{i\theta_3\left(h_2(z_1^{'})\right)}$
$= r_3(z_1^{''})\ e^{i\theta_3(z_1^{''})}, \forall\, z_1^{'} \in V_2$
Now, $R_2(e').e^{i\varphi_1\left(e'\right)} \le R_2(z_1^{'},z_2^{'}).e^{i\varphi_2(z_1^{'},z_2^{'})}$
$= R_3(h_2(e')).e^{i\varphi_2(h_2(e'))}$
$=R_3(h_2(z_1^{'}),h_2(z_2^{'})).e^{i\varphi_2((h_2(z_1^{'}),h_2(z_2^{'}))}$
$$= R_3\left(e^{''}\right)\cdot e^{i\varphi_3\left(e^{''}\right)}, \forall e^{'} = \left(z_1{}^{'},z_2^{'}\right)\in E_2 \qquad (D)$$
From all the above equations, we can write
$r_1\left(z_1\right)\cdot e^{i\theta_1(z_1)} = r_2\left(h_1\left(z_1\right)\right)\cdot e^{i\theta_2(h_1(z_1))}, \forall z_1 \in V_1$
$= r_2(z_1^{'})\ e^{i\theta_2(z_1^{'})}$
$= r_3(h_2(z_1^{'})).e^{i\theta_3\left(h_2(z_1^{'})\right)}$
$= r_3(z_1^{''})\ e^{i\theta_3(z_1^{''})}, \forall\, z_1 \in V_1$
And $R_1\left(e\right).e^{i\varphi_1(e)} \le R_1(z_1,z_2).e^{i\varphi_2(z_1,z_2)}$
$= R_2(h_1(e)).e^{i\varphi_2(h_1(e))}$
$=R_2(h_1(z_1),h_1(z_2)).e^{i\varphi_2(h_1(z_1),h_1(z_2))}$
$=R_2(z_1^{'},z_2^{'}).e^{i\varphi_2(z_1^{'},z_2^{'})}$
$=R_2(e^{'}).\ e^{i\varphi_2(e^{'})}, \forall e=(z_1,z_2)\in E_1$
$=R_3(h_2(z_1^{'}),h_2(z_2^{'})).e^{i\varphi_2(h_2(z_1^{'}),h_2(z_2^{'}))}$
$=R_3(z_1^{''},z_2^{''}).e^{i\varphi_3(z_1^{''},z_2^{''})}$
$=R_3(e'').\ e^{i\varphi_3(e'')}, \forall e=(z_1,z_2)\in E_1$
Therefore $h_2\circ h_1$ is weak isomorphism from $G_{c_1} to\ G_{c_3}$. Hence it is partial order relation.

Theorem 3.10.

If any two complex fuzzy graphs are isomorphic if and only if their complement is isomorphic.

Proof. Let G_{c_1} and G_{c_2} be two complex fuzzy graphs. Assume that $G_{c_1} \cong G_{c_2}$ then there exists a 1-1 and onto mapping $h : V_1 \to V_2$ satisfying $r_1(z_1).e^{i\theta_1(z_1)} = r_2(h(z_1)).e^{i\theta_2(h(z_1))}$

$=r_2(z_1^{'})\ e^{i\theta_2(z_1^{'})}, \forall\ z_1 \in V_1$

And $R_1(e).e^{i\varphi_1(e)} = R_2(h(e)).e^{i\varphi_2(h_1(e))}$

$=R_2(h(z_1),h(z_2)).e^{i\varphi_2(h(z_1),h(z_2))}$

$=R_2(z_1^{'},z_2^{'}).e^{i\varphi_2(z_1^{'},z_2^{'})}$

$=R_2(e^{'}).\ e^{i\varphi_2(e^{'})}, \forall e = (z_1,z_2) \in E_1$

By the definition of complement of CFG,

$\bar{\sigma}_c(z) = \sigma_c(z)$ and

$\mu_c(\bar{z_1,z_2}) = \bar{R}(e)e^{i\bar{\phi}(e)}$, where $\bar{R}(e) = \ min\{r(z_1), r(z_2)\} - R(e)$ and $\bar{\phi}(e) = \ min\{\theta(z_1), \theta(z_2)\} - \phi(e)$, for all $z_1, z_2 \in V$.

$\bar{R_1}(e)e^{\bar{\varphi_1}(e)}= min\{r_1(z_1), r_1(z_2)\}.\ e^{imin\{\theta_1(z_1),\theta_1(z_2)\}}$-$R_1(e).e^{i\varphi_1(e)}$

$=min\{r_2(h(z_1)), r_2(\ h(z_2))\}.\ e^{imin\{\theta_2(h(z_1)),\theta_2(h(z_2))\}}$-$R_1(e).e^{i\varphi_1(e)}$

$=\bar{R'}(e)e^{i\bar{\phi'}(e)}$

(i, e)$\overline{G_{c_1}} \cong \overline{G_{c_2}}$

Conversely, assume that $\bar{G}_{c_1} \cong \bar{G}_{c_2}$, there exists a 1-1 and onto map $h' : V_1 \to V_2$

satisfying $r_1(z).e^{i\theta_1(z_1)} = r_2(h'(z)).e^{i\theta_2(h_1(z_1))}$

$\bar{R_1}(e)e^{\bar{\varphi_1}(e)}= min\{r_1(h'(z_1)), r_1(h'(z_2))\}.\ e^{imin\{\theta_1(h'(z_1)),\theta_1(h'(z_2))\}}$-$R_1(e).e^{i\varphi_1(e)}$

From above $R_1(e).e^{i\varphi_1(e)} = R_2(e').e^{i\varphi_2(e')}$

Hence $h' : V_1 \to V_2$ is an isomorphism from $G_{c_1} to\ G_{c_2}$ (i.e.) $G_{c_1} \cong G_{c_2}$

Theorem 3.11.

If two complex fuzzy graphs $G_{c_1} and\ G_{c_2}$ has weak isomorphism then weak isomorphism exists from $\bar{G}_{c_2}$ to $\bar{G}_{c_1}$

Definition 3.12.

A CFG G_c is said to be self-complementary complex fuzzy graph if $G_c \cong \overline{G_c}$

Theorem 3.13. Let $G_c = (\sigma_c, \mu_c)$ be a self-complementary CFG then$\sum_{z_i \neq z_j} \mu(z_i, z_j) = \frac{1}{2}\sum_{z_i \neq z_j}[r(z_i) + r(z_j)]$for all $z_i, z_j \in V$.

Remarks 3.14. Converse of the theorem 3.6 is not true.

4 Conclusion

This study has introduced the concept of weak isomorphism, co-strong isomorphism, and isomorphism between complex fuzzy graphs. The isomorphism between complex fuzzy graphs is an equivalence relation and weak isomorphism between complex fuzzy graphs is partial order relation is verified. Finally, the self-complementary complex fuzzy graphs are defined. Many theorems may be proved using self-complementary CFG.

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