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* **Corresponding author**.

sathiyap.set@dsuniversity.ac.in, charukanishk@gmail.com

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Homogeneous Quadratic Equation with Four Unknowns $x^2+xy+y^2=z^2+zw+w^2$

R Sathiyapriya[1]*, M A Gopalan[2]

1 Associate Professor, Department of Mathematics, School of Engineering and Technology, Dhanalakshmi Srinivasan University, Trichy, 621112, Tamil Nadu, India
2 Professor, Department of Mathematics, Shrimati Indira Gandhi College, (Affiliated to Bharathidasan University), Trichy, 620002, Tamil Nadu, India

Abstract

Objectives: Diophantine research focuses on various ways to tackle multi variable and multi-degree Diophantine problems. A Diophantine equation is a polynomial equation with only integer solutions. The objective of this manuscript is to find the solutions to Polynomial Diophantine equation $x^2+xy+y^2=z^2+zw+w^2$. **Methods:** Diophantine equations may have finite, infinite, or no solutions in integers. There is no universal method for finding solutions to Diophantine equations. Different choice of solutions in integers is obtained through using linear transformations and employing the factorization method. **Findings:** Many distinct patterns of integer solutions are obtained. **Novelty:** The main thrust is to illustrate different ways of obtaining various choices of solutions in integers to second-degree equations with four variables $x^2+xy+y^2=z^2+zw+w^2$. Different choice of solutions in integers is obtained through using linear transformations and employing the factorization method. Utilization of substitution strategy reduces the given equation to a ternary quadratic equation for which solutions can be found easily.

Mathematics Subject Classification:11D09

Keywords: Homogeneous second degree with four variables; Solutions in integers; Factorization method; Linear transformation; Polynomial diophantine equation

1 Introduction

In the Theory of numbers, Quadratic Diophantine equations play a vital role. There is a wide range of problems in second-degree Diophantine equations. Gopalan MA and Sivakami B[1], introduced the Integral solutions of quadratic with four unknowns in 2012. Gopalan MA, Vidhyalakshmi S, and Thiruniraiselvi N.[2] initiated the homogeneous biquadratic equation with four unknowns in 2015. Gopalan MA[3] introduced the diophantine equation in 2018. Guo L, and Capecelatro J[4] stated the role of clusters on heat transfer in sedimenting gas-solid flows in 2019. Adiga S[5] initiated on bi-quadratic equation with four unknowns in 2020. Premalatha E[6] introduced On Non - homogeneous cubic equation with four unknowns in 2021. Keren D, Osadchy M, Shahar A[7] A Fast and Reliable Solution to PnP, Using Polynomial Homogeneity and a Theorem of Hilbert in 2023.

Mahalakshmi M, Kannan J, Deepshika A, Kaleeswari K[8] Existence and Non-Existence of Exponential Diophantine Triangles Over Triangular Numbers in 2023. Pandichelvi V, Vanaja R, A[9] Paradigm for Two Classes of Simultaneous Exponential Diophantine Equations in 2023.

Janaki, G., & Shankari, A. G[10] Exponential Diophantine Equation. Though there are many second-degree Diophantine equations (homogeneous or non-homogeneous), still it is an important topic of current research. This paper illustrates the process of determining many solutions in integers solutions for the homogeneous second-degree equation with four variables represented by $x^2+xy+y^2=z^2+zw+w^2$. Employing linear transformations, different choices of integer solutions to the considered equation are exhibited.

2 Methodology

Many distinct patterns of integer solutions are obtained. Different choice of solutions in integers is obtained through using linear transformations and employing the factorization method.

The homogeneous quadratic having four variables under consideration is

$$x^2+xy+y^2=z^2+zw+w^2 \tag{2.1}$$

Taking

$$x=u+v, y=u-v, z=p+u, w=p-u, u\neq v\neq p \tag{2.2}$$

in Equation (3.1), it simplifies to

$$v^2+2u^2=3p^2 \tag{2.3}$$

Solving Equation (2.3) through different ways, v,u,p are found. From Equation (2.2), the respective corresponding solutions in integers for Equation (2.1) are found. The procedure to obtain various choices of solutions in integers for Equation (2.1) is as below:

3 Result and Discussion

Illustration 3.1

Assume

$$p=a^2+2b^2 \tag{3.1}$$

Write the integer 3 in Equation (2.3) as

$$3=(1+i\sqrt{2})(1-i\sqrt{2}) \tag{3.2}$$

Substituting Equation (3.1) & Equation (3.2) in Equation (2.3) and using the method of factorization, take

$$v+i\sqrt{2}\,u=(1+i\sqrt{2})(a+i\sqrt{2}\,b)^2 \tag{3.3}$$

Taking the coefficients of corresponding terms in Equation (3.6), we get

$$u=a^2-2b^2+2ab, v=a^2-2b^2-4ab \tag{3.4}$$

Using Equation (3.1) and Equation (3.4) in Equation (2.2), the corresponding integer solutions to Equation (2.1) are as follows

$$x=2a^2-4b^2-2ab, y=6ab$$
$$z=2a^2+2ab, w=4b^2-2ab$$

Note 3.1

In addition to Equation (3.5), the integer 3 may also be expressed as below :

$$3=\frac{(5+i\sqrt{2})(5-i\sqrt{2})}{9}$$
$$3=\frac{(19+i\sqrt{2})(19-i\sqrt{2})}{121}$$

Following the procedure as in Illustration (**3.1**), two more patterns of solutions in integers for Equation (2.1) are found.

Illustration 3.2

Write Equation (2.3) as

$$v^2 + 2u^2 = 3p^2 * 1 \tag{3.5}$$

Consider integer 1 on the R.H.S. of Equation (3.5) as

$$\frac{1 = ((1 + i2\sqrt{2})(1 - i2\sqrt{2}))}{9} \tag{3.6}$$

Using Equation (3.1), Equation (3.2) & Equation (3.9) in Equation (3.6) and using the factorization method consider

$$\frac{v + i\sqrt{2}\,u = ((1 + i\sqrt{2})(1 + i2\sqrt{2})(a + i\sqrt{2}\,b)^2)}{3} \tag{3.7}$$

Taking the coefficients of corresponding terms in Equation (3.7), we have

$$u = a^2 - 2b^2 - 2ab, v = -a^2 + 2b^2 - 4ab \tag{3.8}$$

From Equations (3.1) and (3.8) and Equation (2.2), the respective solutions in integers to Equation (2.1) are given by

$$x = -6ab, y = 2a^2 - 4b^2 + 2ab$$
$$z = 2a^2 - 2ab, w = 4b^2 + 2ab$$

Note 3.2

It is to be observed that the integer 1 in Equation (3.8) is written in general as

$$1 = \frac{\left(2r^2 - s^2 + i\sqrt{2}2rs\right)\left(2r^2 - s^2 - i\sqrt{2}2rs\right)}{(2r^2 + s^2)^2}$$

giving Equation (3.8) when $r = s = 1$.

Illustration 3.3

Rewrite Equation (2.3) as

$$3p^2 = (v^2 + 2u^2\,) * 1 \tag{3.9}$$

Substituting Equations (3.1) and (3.2) &Equation (3.6) in Equation (3.9) and applying factorization, take

$$(1 + i\sqrt{2})(a + i\sqrt{2}b)^2 = (v + i\sqrt{2}\,u)\frac{((1 + i2\sqrt{2}))}{3} \tag{3.10}$$

Taking the coefficients of corresponding terms in Equation (3.10),

$$9u = 3\left[-a^2 + 2b^2 + 10ab\right] \tag{3.11}$$

As solutions in integers are required, taking a by $3A\&b$ by $3B$ in Equation (3.1) & Equation (3.11) and using Equation (2.2), the respective solutions in integers to Equation (2.1) are represented by

$$x = 12A^2 - 24B^2 + 42AB, y = -18A^2 + 36B^2 + 18AB$$
$$z = 6A^2 + 24B^2 + 30AB, w = 12A^2 + 12B^2 - 30AB$$

Illustration 3.4

Rewrite Equation (2.3) as

$$3p^2 - v^2 = 2u^2 \tag{3.12}$$

Assume

$$u = 3a^2 - b^2 \tag{3.13}$$

The integer 2 in Equation (3.12) is

$$2 = (\sqrt{3}+1)(\sqrt{3}-1) \tag{3.14}$$

Substituting Equations (3.13) and (3.14) in Equation (3.12) and applying factorization, take

$$\sqrt{3}p + v = (\sqrt{3}+1)(\sqrt{3}a+b)^2 \tag{3.15}$$

Taking the coefficients of corresponding terms in Equation (3.15), we have

$$v = 3a^2 + b^2 + 6ab, p = 3a^2 + b^2 + 2ab \tag{3.16}$$

From Equations (3.13) and (3.16) and Equation (2.2), the respective solutions in integer to Equation (2.1) are

$$x = 6a(a+b), y = -2b(b+3a), z = 2a(3a+b), w = 2b(a+b)$$

Note 3.3

In addition to Equation (3.14), we have

$$2 = (3\sqrt{3}+5)(3\sqrt{3}-5), 2 = (11\sqrt{3}+19)(11\sqrt{3}-19)$$

Following the procedure as in Illustration (**3.4**), two more patterns of solutions in integers are found.

Illustration 3.5

Rewrite Equation (2.3) as

$$3p^2 - 2u^2 = v^2 * 1 \tag{3.17}$$

Assume

$$v = 3a^2 - 2b^2 \tag{3.18}$$

The integer 1 in Equation (3.17) is

$$1 = (\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2}) \tag{3.19}$$

Substituting Equation (3.18) & Equation (3.19) in Equation (3.17) and employing the method of factorization, consider

$$\sqrt{3}p + \sqrt{2}u = (\sqrt{3}+\sqrt{2})(\sqrt{3}a+\sqrt{2}b)^2 \tag{3.20}$$

Taking the coefficients of corresponding terms in Equation (3.20),

$$u = 3a^2 + 2b^2 + 6ab, p = 3a^2 + 2b^2 + 4ab \tag{3.21}$$

From Equation (3.18) , Equation (3.21) and Equation (2.2) the respective solutions in integer to Equation (2.1) are

$$x = 6a(a+b), y = 2b(2b+3a), z = 6a^2 + 4b^2 + 10ab, w = -2ab$$

Note 3.4

In addition to Equation (3.19), we have

$$1 = \frac{(3\sqrt{3}+\sqrt{2})(3\sqrt{3}-\sqrt{2})}{25}$$

Following the procedure as in Illustration (**3.5**), a different set of solutions in integers to Equation (2.1) is obtained.

4 Choices of transformations & corresponding solutions

It is worth mentioning that, apart from the transformations given by Equation (2.2), there are other choices of transformations to determine the corresponding integer solutions to Equation (2.1).

Transformation 4.1

Taking

$$x=(k+1)v, y=(k-1)v, z=p+q, w=p-q, k\neq 1, p\neq q \tag{4.1}$$

in Equation (4.1), it leads to

$$q^2+3p^2=(1+3k^2)\,v^2 \tag{4.2}$$

Assume

$$v=a^2+3b^2 \tag{4.3}$$

Using Equation (4.3) in Equation (4.2) and applying the factorization method, consider

$$q+i\sqrt{3}\,p=(1+i\sqrt{3}\,k)(a+i\sqrt{3}\,b)^2 \tag{4.4}$$

Taking the coefficients of corresponding terms in Equation (4.4), we have

$$q=a^2-3b^2-6kab, p=k(a^2-3b^2)+2ab \tag{4.5}$$

From Equation (4.3), Equation (4.5) and Equation (4.1) , the respective solutions in integers to Equation (3.1) are

$$\begin{gathered} x=(k+1)\left(a^2+3b^2\right), y=(k-1)\left(a^2+3b^2\right) \\ z=(k+1)\left(a^2-3b^2\right)-2(3k-1)ab \\ w=(k-1)\left(a^2-3b^2\right)+2(3k+1)ab \end{gathered}$$

Note 4.1

Using Equation (4.3) in Equation (4.2) and applying the factorization method, consider

$$\sqrt{3}p+iq=(\sqrt{3}k+i)(\sqrt{3}b+ia)^2$$

Following the process as in Illustration 3.1, the respective solutions in integer to Equation (2.1) are

$$\begin{gathered} x=(k+1)\left(b^2+3a^2\right), y=(k-1)\left(b^2+3a^2\right) \\ z=(k+1)\left(b^2-3a^2\right)-2(3k-1)ab \\ w=(k-1)\left(b^2-3a^2\right)+2(3k+1)ab \end{gathered}$$

Transformation 4.2

Taking

$$x=(1+k)u, y=(1-k)u, z=p+q, w=p-q, k\neq 1, p\neq q \tag{4.6}$$

in Equation (2.1), it leads to

$$3p^2+q^2=(3+k^2)\,u^2 \tag{4.7}$$

Assume

$$u=3a^2+b^2 \tag{4.8}$$

Using Equation (3.5) in Equation (3.4) and applying the factorization method, consider

$$\sqrt{3}\,p+iq=(\sqrt{3}+ik)(\sqrt{3}\,a+ib)^2 \tag{4.9}$$

Taking the coefficients of corresponding terms in Equation (3.6),

$$q = 6ab + k(3a^2 - b^2), p = (3a^2 - b^2) - 2abk \tag{4.10}$$

From Equation (3.5), Equation (3.7) and Equation (3.3), the respective solutions in integer to Equation (2.1) are

$$\begin{gathered} x = (1+k)\left(3a^2+b^2\right), y = (1-k)\left(3a^2+b^2\right) \\ z = 3a^2 - b^2 + 6ab + \left(3a^2 - b^2 - 2ab\right)k \\ w = 3a^2 - b^2 - 6ab + \left(-3a^2 + b^2 - 2ab\right)k \end{gathered}$$

Note 4.2

Using Equation (3.5) in Equation (3.4) and applying the factorization method, consider

$$q + i\sqrt{3}p = (k + i\sqrt{3})(b + i\sqrt{3}a)^2$$

Following the process as in Illustration 3.2, the respective solutions in integers to Equation (2.1) are

$$\begin{gathered} x = (1+k)\left(b^2+3a^2\right), y = (1-k)\left(b^2+3a^2\right) \\ z = (1+k)\left(b^2-3a^2\right) + 2(k-3)ab \\ w = (1-k)\left(b^2-3a^2\right) + 2(k+3)ab \end{gathered}$$

Transformation 4.3

Taking

$$x = u+v, y = u-v, z = p+q, w = p-q, u \neq v, p \neq q \tag{4.11}$$

in Equation (2.1), it gives

$$q^2 - v^2 = 3(u^2 - p^2) \tag{4.12}$$

Case (i):

Taking

$$u = r^2 + s^2, p = r^2 - s^2, r > s > 0 \tag{4.13}$$

in Equation (4.12), one has

$$q^2 - v^2 = 12r^2\, s^2 \tag{4.14}$$

which is satisfied by

$$q = 3r^2\, s^2 + 1, v = 3r^2\, s^2 - 1 \tag{4.15}$$

Using Equation (4.13) and Equation (4.15) in Equation (4.11), the corresponding integer solutions to

Are given by

$$\begin{gathered} x = r^2 + s^2 + 3r^2s^2 - 1, y = r^2 + s^2 - 3r^2s^2 + 1 \\ z = r^2 - s^2 + 3r^2s^2 + 1, w = r^2 - s^2 - 3r^2s^2 - 1 \end{gathered}$$

Note 4.3

In addition to Equation (4.15), Equation (4.14) is also satisfied by

$$\begin{gathered} q = 3r^2 + s^2, v = 3r^2 - s^2 \\ q = r^2 + 3s^2, v = r^2 - 3s^2 \\ q = 4rs, v = 2rs \end{gathered} \tag{4.16}$$

From Equation (4.13),Equation (4.16) and Equation (4.11), the corresponding four patterns of solutions in integers to Equation (4.1) are found.

Case(ii):
Taking

$$u = r^2 + s^2, p = 2rs, r > s > 0 \tag{4.17}$$

in Equation (4.12), one has

$$q^2 - v^2 = 3(r^2 - s^2)^2 \tag{4.18}$$

which is satisfied by

$$q = 2(r^2 - s^2), v = (r^2 - s^2) \tag{4.19}$$

Using Equation (4.17) and Equation (4.19) in Equation (4.11), the respective solutions in integers to Equation (2.1) are

$$x = 2r^2, y = 2s^2$$
$$z = 2rs + 2\left(r^2 - s^2\right), w = 2rs - 2\left(r^2 - s^2\right)$$

5 Conclusion

In this paper, an attempt has been made to obtain many non-zero distinct integer solutions to the homogeneous quadratic Diophantine equation given by $x^2 + xy + y^2 = z^2 + zw + w^2$. As second-degree is plenty, one may search for solutions in integers or other forms of second-degree equations having four or more variables.

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