ISSN (Print): 0974-6846 ISSN (Online): 0974-5645

The Attractors of Fuzzy Super Iterated Function Systems

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Abstract

Objectives: To study the iterated multi-function systems in the framework of fuzziness. **Methods/Statistical Analysis**: The concept of fuzziness is used to define a new class of superfractals as attractors of fuzzy super iterated function systems. **Findings**: The fuzzy super iterated function systems are defined and some existence theorems on fractals in fuzzy metric spaces are established. **Applications/Improvements**: Our results extend and generalize some of the recent results reported in the literature in fuzzy settings.

Keywords: Fractal Space, Fuzzy Metric Space, Fuzzy Contraction, Fuzzy Hutchinson-Barnsley Operator, Fuzzy Superfractal

1. Introduction

The notion of fuzzy set was introduced in 19651 as a mathematical way to represent the imprecision in everyday life. Thereafter, most of the domains of knowledge were explored in the framework of fuzzy sets. Kramosil and Machalek introduced the concept of fuzzy metric space to measure the uncertainty in computing the distance between two objects or sets2. Their concept was modified by defining a Hausdorff and countable topology on the space^{3,4}. Grabiec extended the celebrated Banach contraction principle in fuzzy metric spaces⁵. Several important properties of Hausdorff fuzzy metric on compact sets are studied in⁶. This has enthused the researcher for the fuzzification of fixed point theory and subsequently a number of results appeared in the literature regarding fixed points of maps satisfying some more general contractive conditions in fuzzy and more generalized spaces (see for example^{5,7-15} and several references therein).

The concept of fuzziness is also extended to fractals¹⁰, a new frontier of science, initiated by Mandelbrot. He observed that many of the real world objects are very complex and irregular in nature and thus cannot be described fully by the traditional Euclidean geometry and felt the need of fractal geometry as a powerful

mathematical tool for handling such complex systems. The non-integer dimension, self-similarity and iterative formulation are the prime characterizations of the fractals. After Mandelbrot, fractals are extensively studied in the literature by various authors. The advancement of the computational tools further enriched the domain of the theory and analysis of fractals with diverse applications in almost all branches of sciences and engineering, for details on fractals and its applications, one can refer¹⁶⁻²⁷. An exciting idea of Iterated Function System (IFS) is presented to define and construct fractals as compact invariant subsets of an IFS with respect to the union of contractions in 16,23,28. Generally fractals are generated by two approaches, namely, deterministic and random. A deterministic fractal is obtained by deterministic approach while the random fractal is the result of random approach or chaos game. Many objects in the nature exhibit both the patterns. For modeling such objects the concept of superfractals were proposed^{19,20}. Recently, Hutchinson-Barnsley (HB) operators on fuzzy metric spaces are studied and an analysis on fractals in such spaces is presented^{29,30}. More recently V-variable fractals in metric spaces¹⁹ are invented.

The purpose of this paper is to define and study fuzzy super IFS and obtain some existence results on superfractals in the settings of fuzzy metric.

2. Preliminaries

Some basic concepts and definitions useful in the sequel are given first.

2.1 Definition¹⁶

Let (X, d) be a metric space. A mapping $f: X \rightarrow X$ is a contraction on X if there exist a real number $0 \le \lambda < 1$ such that for all $x, y \in X$.

$$d(f(x) | f(y) \le \lambda d(x, y)$$

Any such number λ is called a contractivity factor for f.

Theorem 2.1¹⁶. A contraction mapping $f: X \to X$ on a nonempty complete metric space (X, d) has a unique fixed point.

2.2 Definition¹⁶

Let (X, d) be a complete metric space and K(X) be the collection of nonempty compact subsets of X. Then the Hausdorff distance between points A and B of K(X) is defined by

 $H_d(A, B) = \max\{d(A, B), d(B, A)\}, \text{ where } d(A, B) = \max\{\min\{d(x, y): y \in B\}: x \in A\}.$

Then $(K(X), H_a)$ is called a Hausdorff space or a fractal space in the sense of Barnsley (see¹⁶).

Theorem 2.2¹⁶. The space $(K(X), H_d)$ is complete whenever (X, d) is a complete metric space.

2.3 Definition¹⁶

Let (X, d) be a complete metric space and $f_n: X \to X$, n = 1, 2, 3, ..., N be contractions with the corresponding contractivity factors λ_n , n = 1, 2, 3, ..., N. Then the system $\{X; f_n, n = 1, 2, 3, ..., N\}$ is called an iterated function system (IFS) in the metric space (X, d) with contractivity factor $\lambda = \max\{\lambda_n: n = 1, 2, 3, ..., N\}$.

Theorem 2.3²³. Let (X, d) be a complete metric space. If $f_n: X \to X$ is a contraction with respect to the metric d for n = 1, 2, ..., N, then there exists a unique non-empty compact subset A of X that satisfies.

$$A = f_1(A) \cup f_2(A) \cup ... \cup f_N(A)$$

The set A is called a self-similar set with respect to $\{f_1, \dots, f_n\}$

$$f_2, ..., f_N$$
}.

2.4 Definition^{28,30}

Let (X, d) be a complete metric space and $\{X; f_n, n = 1, 2, 3, ..., N\}$ be an IFS. The Hutchinson-Barnsley operator (HB operator) of the IFS is a function $F: K(X) \to K(X)$ defined by

$$F(B) = \bigcup_{n=1}^{N} f_n(B)$$
 for all $B \in K(X)$.

Theorem 2.4^{16,28,30}. Let (X, d) be a complete metric space. Let $\{X; f_n, n=1, 2, 3, ..., N\}$ be an IFS and F be the HB operator of the IFS. Then,

- (i) The HB operator F is a contraction mapping on K(X).
- (ii) There exists only one compact invariant set $A_{\infty} \in K(X)$ of the HB operator F called the attractor (or fractal) of IFS or equivalently, F has a unique fixed point namely $A \in K(X)$.
 - (iii) If $B \in K(X)$ then,

$$H_d(A_\infty, B) \le \frac{H_d(B, F(B))}{1 - \lambda}$$

where *F* is the HB operator and A_{∞} is the attractor (or fractal) of the IFS.

2.5 Definition¹⁸ (also See^{4,5})

Let (X, d) be a compact metric space and $(K(X), H_d)$ be the corresponding Hausdorff metric space. Let $\mathcal{K}(X)^V = \underbrace{\mathcal{K}(X) \times \mathcal{K}(X) \times \cdots \times \mathcal{K}(X)}_{V \text{ Times}}$ be a higher order

space that consists of all V-tuples of compact subset of X, then $(\mathcal{K}(X)^V, H_{d^V})$ is a compact metric space with metric $H_{d^V}: \mathcal{K}(X)^V \to R$ defined as

$$H_{d^{V}}(B,C) = \max_{v \in \{1,2,...,V\}} H_{d}(B_{v}, C_{v}) \text{ for all } B, C \in K(H),$$

where $B = (B_1, B_2, ..., B_V)$, $C = (C_1, C_2, ..., C_V)$ and B_V , $C_V \in K(X)$ for $V \in \{1, 2, ..., V\}$.

2.6 Definition¹⁸

Let (X, d) be a compact metric space and $f_l^n: X \to X$, $l = 1, 2, ..., L_n$, n = 1, 2, ..., N be contractive mappings with corresponding contractivity factors. Then the collection of hyperbolic IFSs $\{F_n: n = 1, 2, ..., N\}$ is called as upper IFS and denoted by $\{X; F_1, F_2, ..., F_N\}$,

where $F_n = \{X; f_1^n, f_2^n, \dots, f_{L_n}^n\}$ and $N \ge 1$ is an integer.

2.7 Definition¹⁸

Let (X,d) be a compact metric space. Let $\{X; F_1, F_2, \dots, F_N\}$ be a super IFS and *A* be the set of indices given as

$$\mathcal{A} = \left\{ a = (n, v) : n = (n_1, n_2, ..., n_V), v = (v_{1,1}, v_{1,2}, ..., v_{1,L_{n_1}}, v_{2,1}, v_{2,2}, ..., v_{2,L_{n_2}}, \cdots, v_{V,1}, v_{V,2}, ..., v_{V,L_{n_V}}), v_{v,l} \in \{1, 2, \cdots, V\}, \right.$$
for $l = 1, 2, \cdots, L_{n_v}, n_v = 1, 2, \cdots, N$ and $v = 1, 2, \cdots, V$.

Then for each, $a \in \mathcal{A}$ $f^a: K(X)^V \to K(X)^V$ is given by $f^{a}(B_{1}, B_{2}, \dots, B_{V}) = \left(\bigcup_{l=1}^{L_{n_{1}}} f_{l}^{n_{1}}(B_{v_{1,l}}), \bigcup_{l=1}^{L_{n_{2}}} f_{l}^{n_{2}}(B_{v_{2,l}}), \dots, \bigcup_{l=1}^{L_{n_{V}}} f_{l}^{n_{V}}(B_{v_{V,l}})\right),$ where $(B_1, B_2, ..., B_V) \in K(X)^V$.

Using these transformations, a V-variable IFS can be defined as $F^{(V)} = {\mathcal{K}(X)^V; f_a, a \in \mathcal{A}}$

Theorem 2.5¹⁶. Let (X, d) be a compact metric space and $(K(X), H_d)$ be the corresponding Hausdorff metric space. If $A, B, C, D \subset K(X)$, then

$$H_d(A \cup B, C \cup D) \leq \max\{H_d(A, C), H_d(B, D)\}.$$

Theorem 2.6¹⁸. Let the IFSs F_n has contractivity factor $\lambda \in$ [0, 1) for all n = 1, 2, 3, ..., N. Then the mapping f^a : K(X) $V \to K(X)^V$ is contractive, with contraction factor λ for all $a \in \mathcal{A}$.

Theorem 2.7¹⁸. $F^{(V)}$ is a hyperbolic IFS, for all V = 1, 2, 3,

Theorem 2.8¹⁸. Let (X, d) be a compact metric space and $F^{(V)}$ a hyperbolic IFS on $K(X)^V$. Then, there exists only one compact invariant set $A^{(V)} \in K(X)^V$, known as *V*-variable super fractal associated with $F^{(V)}$.

In the following, we present some basic concepts in fuzzy metric spaces required for our results. We follow^{3,5}, 8, 30 etc. for the same.

2.8 Definition³

A binary operation *: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if ([0, 1], *) is a topological monoid with unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$ (a, b, c, $d \in [0, 1]$).

2.9 Definition³

The 3-tuple (X, M, *) is said to be a fuzzy metric space

if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, +\infty)$ satisfying the following conditions:

- (i) M(x, y, t) > 0,
- (ii) M(x, y, t) = 1 if and only if x = y,
- (iii) M(x, y, t) = M(y, x, t),
- (iv) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
- (v) $M(x, y, \cdot)$: $(0, +\infty) \rightarrow [0, 1]$ is continuous for all $x, y, +\infty$ $z \in X$ and t, s > 0.

Lemma 2.1³. M(x, y, t) is non decreasing for all x, y in X.

2.10 Definition³

Let (X, d) be a metric space. Define $a * b = a \cdot b$, the usual multiplication for all $a, b \in [0, 1]$, and let M_d be the function defined on $X \times X \times (0, +\infty)$ by $M_d(x, y, t) = \frac{t}{t + d(x, y)}$

Then $(X, M_a, *)$ is a fuzzy metric space called standard fuzzy metric space and $(M_p, *)$ is the fuzzy metric induced by d.

Let (X, M, *) be a fuzzy metric space. An open ball $B_{M}(x, r, t)$ with center $x \in X$ and radius $r \in (0, 1)$ for t > 0is defined as the set

$$B_{M}(x, r, t) = \{ y \in X: M(x, y, t) > 1 - r \}.$$

 $:= \{A \subseteq X : \text{for each } x \in A \text{ there exists} \}$ Define an $r \in (0, 1)$, with $B(x, r, t) \subset A$.

Moreover, for each $x \in X$ the collection of open balls $\{B_{M}(x, x)\}$ 1/n, 1/n): n = 2, 3...}, is a local base at x with respect to τ_M . It is clear that for any fuzzy metric space (X, M, *), τ_M . is a first countable topology (for more details see⁷). Then $\tau_{\scriptscriptstyle M}$ is called the topology generated by the fuzzy metric space (X, M, *).

2.11 Definition^{4,7}

In a fuzzy metric space (X, M, *)

- (i) A sequence $\{x_n\}$ converges to a point $x \in X$ with respect to τ_M iff, $\lim M(x, x_n, t) = 1$ for all t > 0.
- (ii) A sequence $\{x_n\}$ is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and t > 0 there exists $n_0 \in N$ such that $M(x_0)$ $x_{...}, t) > 1 - \varepsilon$ for all $n, m > n_0$.
- (iii) A fuzzy B-contraction is a self-mapping f on Xsuch that $M(f(x), f(y), \lambda t) \ge M(x, y, t)$ for all $x, y \in X$ and t > 0, where $0 < \lambda < 1$. The number λ is then called a *B*-contraction constant of *f*.

2.12 Definition³

A fuzzy metric space in which every Cauchy sequence is convergent is called a complete fuzzy metric space.

Theorem 2.98. A fuzzy metric space is pre-compact if every sequence has a Cauchy sub sequence.

Theorem 2.10⁸. A fuzzy metric space is compact if it is pre-compact and complete.

Grabiec⁵ obtained the fuzzy version of Banach contraction principle in the following manner.

Theorem 2.11⁵. Let (X, M, *) be a complete fuzzy metric space such that $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$. Let $f: X \to X$ is a fuzzy B-contraction on X with contraction factor λ . Then f has a unique fixed point.

2.13 Definition⁶

Let (X, M, *) be a fuzzy metric space and τ_M be the topology induced by the fuzzy metric M. Let K(X) is the set of nonempty compact subsets of X. Then, the Hausdorff fuzzy metric $(H_M, *)$ is a function

$$H_{M}: K(X) \times K(X) \times (0, +\infty) \rightarrow [0, 1]$$
 defined as

$$H_{M}(A,B,t) = \min\{M(A,B,t) \ M(B,A,t)\},$$
 where $M(A,B,t) = \inf_{x \in A} \{\sup_{y \in B} M(x,y,t)\}$.

Theorem 2.12⁶. Let (X, M, *) be a fuzzy metric space. Then

- (i) $(K(X), H_M, *)$ is complete if (X, M, *) is complete.
- (ii) (K(X), H_M, *) is pre-compact if and only if (X, M,
 *) is pre-compact.

2.14 Definition³⁰

Let (X, M, *) be a fuzzy metric space and $f_n: X \rightarrow X$, n = 1, 2, 3, ..., N be N fuzzy B-contraction mappings. Then the system $\{X; f_n, n = 1, 2, 3, ..., N\}$ is called a fuzzy iterated function system (FIFS) of fuzzy B-contractions on the fuzzy metric space (X, M, *).

2.15 Definition³⁰

Let (X, M, *) be a fuzzy metric space. Let $\{X; f_n, n = 1, 2, 3, ..., N\}$ be an FIFS of fuzzy *B*-contractions. Then the Fuzzy Hutchinson-Barnsley (FHB) operator of the FIFS is a function $F: K(X) \to K(X)$ defined by

$$F(B) = \bigcup_{n=1}^{N} f_n(B) \text{ for all } B \in K(X).$$

Theorem 2.13³⁰. Let (X, M, *) be a complete metric space. Let $(K(X), H_M, *)$ be the corresponding Hausdorff fuzzy metric space. Suppose $f_n: X \rightarrow X, n = 1, 2, 3, ..., N$ is a fuzzy B-contraction on (X, M, *). Then the FHB operator is also a fuzzy B- contraction on $(K(X), H_M, *)$.

Lemma 2.2³⁰. Let (X, M, *) be a fuzzy metric space. Let $(K(X) H_M, *)$ be the corresponding Hausdorff fuzzy metric space. If $A, B, C, D \subset X$, then $H_M(A \cup B, C \cup D, t) \ge \min\{H_M(A, C, t) H_M(B, D, t\}$ for every t > 0.

2.16 Definition³⁰

Let (X, M, *) be a complete metric space and $\{X; f_n, n = 1, 2, 3, ..., N\}$ be an FIFS of fuzzy B-contractions and F be the FHB operator of the FIFS. We say set $A_{\infty} \in K(X)$ is a fuzzy attractor or fuzzy fractal of the given FIFS, if A_{∞} is a unique fixed point of the fuzzy HB operator F. Usually such $A_{\infty} \in K(X)$ is also called as a fractal generated by the FIFS of fuzzy B-contractions and so called as an FIFS fractal of fuzzy B-contractions.

3. Main Results

We obtain the fuzzy version of *V*-variable space by using the notion of Hausdorff fuzzy metric space⁶ given by Rodriguez-Lopez and Romaguera (cf. section 5). First a *V*-variable fuzzy fractal space is defined in the following manner.

3.1 Definition

Let (X, M, *) be a compact fuzzy metric space and $(K(X), H_M, *)$ be an associated Hausdorff fuzzy metric space. Let $\mathcal{K}(X)^V = \underbrace{\mathcal{K}(X) \times \mathcal{K}(X) \times \cdots \times \mathcal{K}(X)}_{\text{VTimes}}$ be a higher order

space that consists of all *V*-tuples of compact subset of fuzzy set *X*. Then $(\mathcal{K}(X)^V, H_{M^V}, *)$ is a fuzzy metric space with metric $H_{M^V}: \mathcal{K}(X)^V \times \mathcal{K}(X)^V \times (0, +\infty) \to (0, 1)$ defined as $H_{M^V}(B, C, t) = \min_{v \in \{1, 2, ..., V\}} H_M(B_v, C_v, t)$ for all $B, C \in K(H)$ and t > 0, where $B = (B_1, B_2, ..., B_V)$, $C = (C_1, C_2, ..., C_V)$ and $B_v, C_v \in K(X)$ for all $v \in \{1, 2, ..., V\}$.

Theorem 3.1. Let (X, M, *) be a fuzzy metric space. Then $(K(X), H_M, *)$ is compact if and only if (X, M, *) is compact.

Proof: Since, by theorem 2.12(ii), $(K(X), H_{M}, *)$ is precompact if and only if (X, M, *) is pre-compact and by theorem 2.12(i), $(K(X), H_M, *)$ is complete if and only if $(X, Y_M, *)$ M, *) is complete. Therefore, $(K(X), H_M, *)$ is pre-compact and complete if and only if (X, M, *) is pre-compact and complete. Using theorem 2.10, we deduce that $(K(X), H_{M})$ *) is compact if and only if (X, M, *) is compact.

Theorem 3.2. Let $(K(X), H_M, *)$ be a complete fuzzy metric space. Then $(\mathcal{K}(X)^{V}, H_{M^{V}}, *)$ is complete fuzzy metric space if and only if $(K(X), H_{M}, *)$ is complete.

Proof. Let $(K(X), H_M, *)$ be a pre-compact fuzzy metric space. Also assume $\{A_n\}_{n\in\mathbb{N}}$ to be a Cauchy sequence in K(X)V i.e. if $A_n = (A_1^n, A_2^n, ..., A_V^n)$ $A_m = (A_1^m, A_2^m, ..., A_V^m)$, then for each $\varepsilon \in (0, 1)$ and t > 0, there exists $n_0 \in N$ (set of natural numbers) such that, for all $n, m \ge n_0$

$$\begin{split} &H_{M^{V}}(A_{n},A_{m},t)>1-\varepsilon,\text{i.e.,}\\ &H_{M^{V}}\Big((A_{1}^{n},A_{2}^{n},...,A_{V}^{n})\;\;(A_{1}^{m},A_{2}^{m},...,A_{V}^{m})\;\;t\Big)>1-\varepsilon\\ &\min\{H_{M}(A_{1}^{n},A_{1}^{m},t)\;\;H_{M}(A_{1}^{n},A_{2}^{m},t)\;\;...,H_{M}(A_{V}^{n},A_{V}^{m},t)\}\;\;>1-\varepsilon. \end{split}$$

Thus, we have $H_M(A_i^n, A_i^m, t) > 1 - \varepsilon$ for all $n, m \ge n_0$, j = 1, 2, ..., V.

Therefore, $\{A_j^n\}_{n\in\mathbb{N}}$ for j=1, 2, ..., V are Cauchy sequences in K(X). Since $(K(X), H_M, *)$ is a complete metric space, there exists an $A_i \in K(X)$ such that

$$H_M(A_i^n, A_i^m, t) \to \infty \text{ as } n \to \infty \text{ for all } t > 0.$$

Let $A = (A_1, A_2, ..., A_N)$, then $A \in K(X)^V$ implies that $H_M(A_n, A, t) \to 1$ as $n \to \infty$ for all t > 0. Which shows $(\mathcal{K}(X)^{V}, H_{M^{V}}, *)$ is a complete fuzzy metric space.

The converse of the theorem can be proved similarly by following above steps in the reverse order.

Theorem 3.3. Let $(K(X), H_M, *)$, be a pre-compact fuzzy metric spaces, then $(\mathcal{X}(X)^{\vee}, H_{M^{\vee}}, *)$ is precompact fuzzy metric space if and only if $(K(X), H_M, *)$ is pre-compact.

Proof. Let $(K(X), H_M, *)$ be a pre-compact fuzzy metric space. Also assume a sequence $\{A_n\}_{n\in\mathbb{N}}$ in $K(X)^V$ i.e. if $A_n = (A_1^n, A_2^n, ..., A_V^n)$, then $\{A_j^n\}_{n \in \mathbb{N}}$ for j = 1, 2, ..., V is a sequence in K(X). Since $(K(X), H_M, *)$ is pre-compact, there exists Cauchy subsequences $\{A_i^{n_k}\}_{n\in\mathbb{N}}\in\mathcal{K}(X)$ for j= 1, 2, ..., V. Therefore, if $\{A_{n_k} = (A_1^{n_k}, A_2^{n_k}, ..., A_V^{n_k})\}_{n \in \mathbb{N}}$

, then $\{A_{n_k}\}_{n\in\mathbb{N}}$ is Cauchy subsequence of $\{A_n\}_{n\in\mathbb{N}}\in K(X)$ V. Which shows $(\mathcal{K}(X)^V, H_{M^V}, *)$ is a pre-compact fuzzy metric space.

Following reverse steps one can easily prove the converse of the theorem.

Theorem 3.4. Let (X, M, *) be a fuzzy metric space. Then, $(\mathcal{K}(X)^V, H_{M^V}, *)$ is compact if and only if (X, M, *) is compact.

Proof. Using Theorem 2.10, 3.2 and 3.3, we can easily deduce that is compact if and only if $(\mathcal{K}(X)^{V}, H_{M^{V}}, *)$ is compact. Further, by theorem 3.1, $(K(X), H_M, *)$ is compact if and only if (X, M, *) is compact. Therefore, from above two arguments it can be deduced that $(\mathcal{K}(X)^{\vee}, H_{\mathcal{A}^{\vee}}, *)$ is compact if and only if (X, M, *) is compact.

In the following we define a fizzy super IFS and a *V*-variable fuzzy IFS.

3.2 Definition

Let (X, M, *) be a compact fuzzy metric space and $f_l^n: X \to X$ for $l = 1, 2, ..., L_n$ be fuzzy B-contraction mappings with same contraction factor λ . Then the collection of hyperbolic fuzzy IFSs $\{F_n : n = 1, 2, ..., N\}$ where $F_n = \{X; f_1^n, f_2^n, \dots, f_{L_n}^n\}$ and $N \ge 1$ is an integer, are said to be a fuzzy super IFS, denoted by $\{X; F_1, F_2, \dots, F_N\}.$

3.3 Definition

Let (X, M, *) be a compact fuzzy metric space and $\{X; F_1, F_2, \dots, F_N\}$ a Fuzzy super IFS. Let A denotes the set of indices

$$\mathcal{A} = \left\{ a = (n, v) : n = (n_1, n_2, ..., n_V), \quad v = (v_{1,1}, v_{1,2}, ..., v_{1,L_m}, v_{2,1}, v_{2,2}, ..., v_{2,L_{n_2}}, \cdots, v_{V,1}, v_{V,2}, ..., v_{V,L_{m_V}}), \quad v_{v,l} \in \{1, 2, \cdots, V\} \right\}$$
for $l = 1, 2, \cdots, L_{n_v}, n_v = 1, 2, \cdots, N$ and $v = 1, 2, \cdots, V$.

For each $a \in \mathcal{A}$ and $(B_1, B_2, ..., B_V) \in K(X)^V$, $f^a: K(X)$ $V \to K(X)^V$ is given by

$$f^{a}(B_{1}, B_{2}, \dots, B_{V}) = \left(\bigcup_{l=1}^{L_{n_{1}}} f_{l}^{n_{1}}(B_{v_{1,l}}), \bigcup_{l=1}^{L_{n_{2}}} f_{l}^{n_{2}}(B_{v_{2,l}}), \dots, \bigcup_{l=1}^{L_{n_{V}}} f_{l}^{n_{V}}(B_{v_{V,l}})\right).$$

Then a V-variable fuzzy IFS is defined by the transformations $F^{(V)} = {\mathcal{K}(X)^V; f_a, a \in \mathcal{A}}.$

Theorem 3.5. Let the fuzzy IFSs F_n has contraction factor $\lambda \in (0, 1)$ for all n = 1, 2, 3, ..., N. Then the mapping $f^a:K(X)^V \to K(X)^V$ is a fuzzy B-contraction, with same contraction factor λ for all $a \in A$.

Proof. For v = 1, 2, ..., V and all $B = (B_1, B_2, \cdots, B_{L_{n_v}})$ and $C = (C_1, C_2, \cdots, C_{L_n})$, we have

$$\begin{split} H_{M}\left(\bigcup_{l=1}^{L_{n_{v}}}f_{l}^{n_{v}}(B_{l}),\bigcup_{l=1}^{L_{n_{v}}}f_{l}^{n_{v}}(C_{l}),\lambda t\right) \\ &=H_{M}\left(f_{1}^{n_{v}}(B_{1})\bigcup f_{2}^{n_{v}}(B_{2})\bigcup\cdots\bigcup f_{L_{n_{v}}}^{n_{v}}(B_{L_{n_{v}}}),\\ &f_{1}^{n_{v}}(C_{1})\bigcup f_{2}^{n_{v}}(C_{2})\bigcup\cdots\bigcup f_{L_{n_{v}}}^{n_{v}}(C_{L_{n_{v}}}),\lambda t\right) \\ &\geq \min_{l\in\{1,2,\dots,L_{n_{v}}\}}\left\{H_{M}\left(f_{l}^{n_{v}}(B_{l}),f_{l}^{n_{v}}(C_{l}),\lambda t\right)\right\} \\ &\text{(by Lemma 2.2)} &\geq \min_{l\in\{1,2,\dots,L_{n_{v}}\}}\left\{H_{M}(B_{l},C_{l},t)\right\},\\ &\geq H_{M^{L_{n_{v}}}}\left((B_{1},B_{2},\dots,B_{L_{n_{v}}}),(C_{1},C_{2},\dots,C_{L_{n_{v}}}),t\right) \\ &= H_{M^{L_{n_{v}}}}\left(B,C,t\right). \end{split}$$

Hence, for all B_1 , B_2 , ..., B_V and C_1 , C_2 , ..., $C_V \in K(X)$, we obtain

$$\begin{split} H_{M^{r}}(f^{a}(B_{1},B_{2},\cdots,B_{V}),f^{a}(C_{1},C_{2},\cdots,C_{V}),\lambda t) \\ &= \min_{v \in \{1,2,\dots,V\}} \left\{ H_{M}\left(\bigcup_{l=1}^{L_{n_{v}}} f_{l}^{n_{v}}(B_{l}),\bigcup_{l=1}^{L_{n_{v}}} f_{l}^{n_{v}}(C_{l}),\lambda t\right) \right\} \\ &\geq \min_{v \in \{1,2,\dots,V\}} \left\{ H_{M^{L_{n_{v}}}}\left((B_{v,1},B_{v,2},\cdots,B_{v,L_{n_{v}}}),(C_{v,1},C_{v,2},\cdots,C_{v,L_{n_{v}}}),t\right) \right\} \\ &\geq H_{M^{r}}\left((B_{1},B_{2},\cdots B_{V}),(C_{1},C_{2},\cdots C_{V}),t\right). \end{split}$$

This completes the proof.

Theorem 3.6. The $F^{(V)}$ is a hyperbolic IFS, for all V = 1, 2, 3, ...

Proof. We have $F^{(V)} = \{ {}^{\alpha}K(X)^V; f^a, a \in {\mathcal A} \}$ From theorem 3.5, f^a is a fuzzy B-contraction map on $K(X)^V$. Since f^a is a fuzzy B-contraction map on $K(X)^V$, therefore $F^{(V)}$ is also a B-contraction map on $K(X)^V$. It follows that $F^{(V)}$ is a hyperbolic fuzzy IFS, for all $V = 1, 2, 3, \ldots$.

Theorem 3.7. Let $(X, M, {}^*)$ be a compact metric space and $F^{(V)}$ be a fuzzy super IFS. Then, there exists a unique compact invariant set $A^{(V)} \in K(K(X)^V)$ or, equivalently, $F^{(V)}$ has a unique fixed point namely $A^{(V)} \in K(K(X)^V)$.

Proof. Since (X, M, *), is a compact fuzzy metric space, therefore by theorem 3.2, $(\mathcal{K}(X)^{V}, H_{M^{V}}, *)$ is also

a compact Hausdorff fuzzy metric space. Further, by theorem 3.6, the *V*-variable fuzzy super IFS are a fuzzy *B*-contraction. Then by theorem 2.11, we conclude that $F^{(V)}$ has a unique fixed point, namely $A^{(V)} \in K(K(X)^V)$.

The following experiment illustrates the result of theorem 3.7; when V=2.

4. Experiment

Assume $F_1 = \{X \times X; f_1^1, f_2^1\}$ and $F_2 = \{X \times X; f_1^2, f_2^2\}$, then the corresponding fuzzy super IFS denoted as $\{X \times X; F_1, F_2\}$. Also take two digital buffers of images $(\mathbf{I}_1, \mathbf{I}_2)$ and $(\mathbf{O}_1, \mathbf{O}_2)$ of same size known as input and output buffers. Now, construct a sequence of pairs of images using following steps (see Barnsley et al^{17 p. 389}):

- Randomly pick one of the IFSs F₁ or F₂, say Fₙ₁ (n₁∈{1, 2}). Apply f₁ⁿ₁ to the randomly selected image (I₁ or I₂), to make an image on O₁. Then apply f₂ⁿ₁ to the image on either I₁ or I₂, also selected randomly and overlay the resulting image on the image now already on O₁.
- Again pick randomly one of the IFSs F₁ or F₂, say F₂
 (n₂∈ {1, 2}). Apply f₁²₂ to the randomly selected image (I₁ or I₂), to make an image on O₂. Also apply f₂²² to the image on I₁ or I₂, also selected randomly and overlay the resulting image on the image now already on O₂.
- Switch input and output, clear the new output screens and repeat steps (i) and (ii).
- Repeat step (iii) many times, to allow the system to settle into its stationary state.

Example 3.1. Let X = [0, 1], $a * b = a \cdot b$ for all $a, b \in [0, 1]$ and $\varphi(t) = t/(t+1)$. Obviously, (X, M, *) is a complete fuzzy metric space, with fuzzy metric

$$M(x, y, t) = [\varphi(t)]^{|x-y|}$$
 for all $x, y \in X$.
We consider the following maps on $X \times X$:

$$f_1^1(x,y) = (x/2+y/3+3/16, x/2-y/3+5/16),$$

$$f_2^1(x,y) = (x/2-y/3+5/16, -x/2-y/3+13/16),$$

$$f_1^2(x,y) = (x/2+y/3+3/16, -x/2+y/3+11/16),$$

$$f_2^2(x,y) = (x/2-y/3+5/16, x/2+y/3+3/16).$$

The maps defined above are contractive over the given fuzzy metric space $(X \times X, M, *)$, where $\mathcal{M}((x_1, y_1), (x_2, y_2), t) = \min\{M(x_1, x_2, t), M(y_1, y_2, t)\}$. For the square initiator $[0, 1] \times [0, 1]$ seen in Figure 1, the

set attractors for F_1 and F_2 after 6, 9, 12 and 15 iterations are depicted in the Figure 2, Figure 3, Figure 4 and Figure 5 respectively.

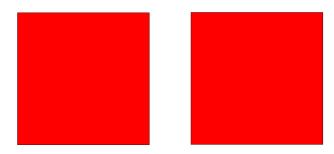


Figure 1. Initial Images I_1 and I_2 .

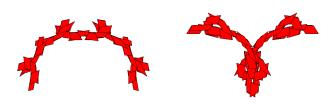


Figure 2. Buffers of 2-Variable Fuzzy IFS for N = 6.



Figure 3. Buffers of 2-Variable Fuzzy IFS for N = 9.



Figure 4. Buffers of 2-Variable Fuzzy IFS for N = 12.



Figure 5. Buffers of 2-Variable Fuzzy IFS for N = 15.

5. Conclusion

The iterated multi-function systems and fuzzy super iterated function systems are studied in a new setting. Some existence and uniqueness theorems generalizing some of the recent results reported in the literature are obtained.

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