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A Taxonomy on Rigidity of Graphs

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Abstract

Objective: In the material world, objects like railway tracks, bridges, roofs etc., are constructed by collections of non-elastic rigid rods, beams, etc.. A structure is said to be rigid if there is no continuous motion of the structure that changes its shape without changing the shapes of its components like rods or beams. In this survey work, we accumulate the fundamental concepts on graph rigidity. **Methods and Analysis:** We give the analytical definition of rigid graphs using the idea of rigid motions. The Laman's theorem and Hendrickson's algorithm are presented as methods for testing graph rigidity in the plane. The construction of the rigid graphs is also analyzed using the Henneberg's operations. We describe how in distributed environments the rigidity of graphs can be checked using the vertex ordering in the graph. For frameworks lying in the higher dimensional spaces rigidity testing method is presented in form of a theorem. **Novelty and Improvements:** The results of this review works may help the readers to better understand the graph rigidity theory from different perspectives. This study founds a positive association between the analytical and combinatorial concepts of graph rigidity investigated so far.

Keywords: Graph Realization, Localizability Testing, Network Localization, Rigidity of Graphs

1. Introduction

Rigidity prevents the deflection of each solid structure under externally applied forces. In real life applications, constructions like bridges and roofs, etc. (Figure 1) is made of rigid rods hinged at their end points. Such a structure may be viewed as *bar-joint framework* which may bend at the joints. Thus it is crucial to prevent the bending or flexing of bar-joint structures under external forces. The number of rods may be increased to enhance the rigidity of the structure. A bar-joint structure may be modeled as an edge-weighted graph with joints as vertices, bars as edges, and lengths of the bars as edge-weights. Rigidity of such edge-weighted graphs is covered in the graph rigidity theory.

Figure 1. Constructions of bridges and roofs[15].

The notion of graph rigidity first appeared in Cauchy's rigidity theorem for a convex triangulated polyhedron with rigid edges and flexible joints[1,2]. In the nineteenth century the rigidity theory was developed for the *bar*

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and joint structures[3]. Characterizing the flexibility of constructions like roads, railways, airlines, wireless sensor networks etc., by using their graphical representations in the Euclidean space is given special attention in different domains of studies such as material science, engineering, etc. Recently it acquires much attention due to its growing applications in different fields, e.g., in transportation problem, VLSI design, robotics and social networks.

1.1 Graph Realization in the Plane

The real life objects like roofs, bridges or sensor networks can be represented as graph G = (V, E), where the set of bars can be identified by edge set E and the set of joints by vertex set V. Since multiple edges between same pair of joints has no effect on equilibrium of the structure, without lose of generality, we put G to be a simple graph. The study of the structural rigidity with the help of underlying graph G needs reconstruction of the structure from G. This reconstruction problem can be modeled as the graph realization problem[4].

A realization of the graph G = (V, E) in the plane is a one to one mapping $p: V \to R^2$, i.e., every vertex $v \epsilon V$ is assigned a position $p(v) \epsilon R^2$. For any p, (G,p) is called a *configuration* of G. The pair is also called a framework or a straight line realization of the graph in R^2. If $v \epsilon V$ then $p(v)$ is a joint of the framework. If $\{u, v\} \epsilon E$ then the straight line joining p(u) and p(v) is called a *bar* of the framework (G, p). Let, $|V| = v$, then $p(V)$ is a point in R^{2v}. Figure 2 shows three different frameworks (a), (b), (c) for the same graph.

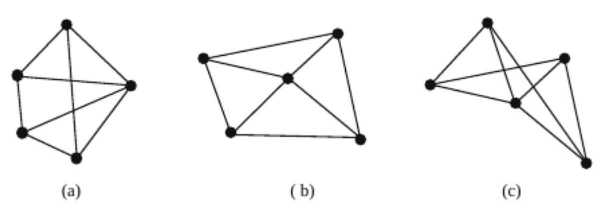


Figure 2. Different frameworks for a graph.

Let (G, p) and (G, q) be two different realizations of a graph *G(V, E)* in R^2. They are said to be $equivalent$ if $\|p(u) - p(v)\| = \|q(u) - q(v)\|$ for all edges $\{u, v\} \epsilon E$ (Here $\|.\|$ is the standard Euclidean norm in R^2). If this equality holds for all pairs of vertices of the graph G then (G, p) and (G, q) are said to be congruent. (G, p) is called a unique realization of G if all the equivalent frameworks are congruent. Figure 3(a) can be obtained from Figure 3(b) by reflecting the sub graph below the edge e_5 with respect to e_5. Therefore these two frameworks are equivalent but not congruent.

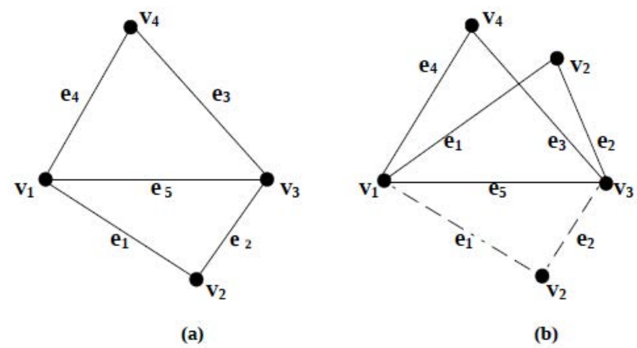


Figure 3. Equivalent frameworks which are not congruent.

In Figure 4(a), two equivalent frameworks of a quadrilateral are shown which are not congruent. However, Figure 4(b) has a unique realization up to congruence.

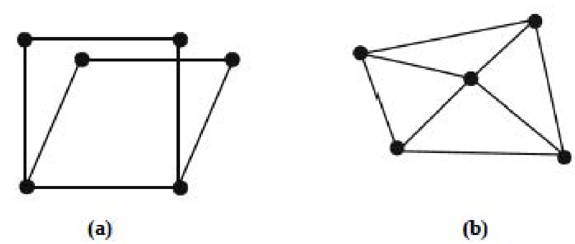


Figure 4. (a) is not uniquely realizable but (b)

1.2 Edge Function of a Graph

Different realizations of a graph G may have different lengths between the terminal points of a particular edge in E. Figure 2 shows different realizations of a graph G having different edge lengths. In reality, if G(V,E) represents a bar-joint structure then any realization of G preserves the edge lengths. If $p = (t_1, t_2, \ldots, t_v)$ is a realization of G in R^2 then p may be considered as a point in R^{2v} because each $t_i \in R^2$. With respect to the realization p, each edge $e_r = \{v_i, v_j\}$ is assigned a value (denote by $\| e_r \|$) where $\| e_r \| = \| t_i - t_j \|$ and $\|.\|$ is the Euclidean norm.

Definition 1: Let $G(V; E)$ be a graph having vertex set V labelled $v_1, v_2, \ldots, v_v$ and edge set E labelled as $e_1, e_2, \ldots, e_k$. Suppose each vertex v_i is assigned a position t_i for $1 \leq i \leq v$. A function $f_G: R^{2V} \to R^k$ defined by $f_G(t_1, t_2, \ldots, t_v)$: = (|| e_1 ||^2, || e_2 ||^2,…, || e_k ||^2) is called the *edge function*[2] of G .

Intuitively, for $p = (t_1, t_2, \ldots, t_v) \in R^{2V}$ if $(G;p)$ is a framework in R^2 then $f_G(\mathbf{p}) \epsilon R^k$ gives the squares of the edge lengths of $(G;p)$ in the given order.

Example 1. In Figure 5, the rectangle has vertex set $V = \{v_1, v_2, v_3, v_4\}$ and edge set $E = \{e_1, e_2, e_3, e_4, e_5\}$ where,

$e_1 = \{v_1, v_2\}$;

$e_2 = \{v_2, v_3\}; e_3 = \{v_3, v_4\}; e_4 = \{v_4, v_1\}; e_5 = \{v_1, v_3\};$

Let $t_1 = (0;0)$; $t_2 = \left(\frac{2}{3}; -\frac{1}{3}\right); t_3 = (1;0); t_4 = \left(\frac{1}{3}; \frac{2}{3}\right)$

be the positions of vertices v_1, v_2, v_3, v_4 respectively in Figure 5(a). The square of edge lengths are,

|| e_1 ||^2 = 5/9, || e_2 ||^2 = 2/9, || e_3 ||^2 = 8/9, || e^4 ||^2 =5/9, || e^5 ||^2 = 1;

Therefore,

$$f_G(t_1, t_2, t_3, t_4) = \left(\frac{5}{9}, \frac{2}{9}, \frac{8}{9}, \frac{5}{9}, 1\right).$$

If $t_1 = (0;0)$; $t_2 = \left(\frac{2}{3}; \frac{1}{3}\right); t_3 = (1;0); t_4 4 = \left(\frac{1}{3}; \frac{2}{3}\right)$ in Figure 5(b) then,

$$f_G(t_1, t_2, t_3, t_4) = \left(\frac{5}{9}, \frac{2}{9}, \frac{8}{9}, \frac{5}{9{,}1}\right).$$

Eventually in this case two realizations are equivalent but not congruent.

Let $(G;p)$ be a realization of $G(V;E)$ in the plane. There may be many different realizations q of G such that $f_G(p) = f_G(q)$. The set $f_G^{-1}(f_G(p)) = \{x : x \in R^{2v}$ and $f_G(p) = f_G(q)\}$ is called the *fiber of* G for the realization $(G;p)$ and is denoted by $Fiber(G; p)$. $Fiber(G;p)$ is the set of all equivalent realizations of $(G;p)$. For the graph $G = (V;E)$, let $|V| = \mathcal{V}$ and K_V be the complete graph with the same vertex set V . For $, q \in R^{2V}$, $f_{K_V}(p) = f_{K_V}(q)$ if and only if the frameworks (K_V ;p) and (K_V ;q) are congruent. $Fiber(K_V; p)$ is the set of all congruent realizations of $(G;p)$. Every realization of K_V congruent to $(K_V; p)$ gives a realization of G congruent to $(G;p)$. In view of the property that every congruent realization of $(G;p)$ is an equivalent realization of $(G;p)$, the above discussion may be summarized in the following lemma.

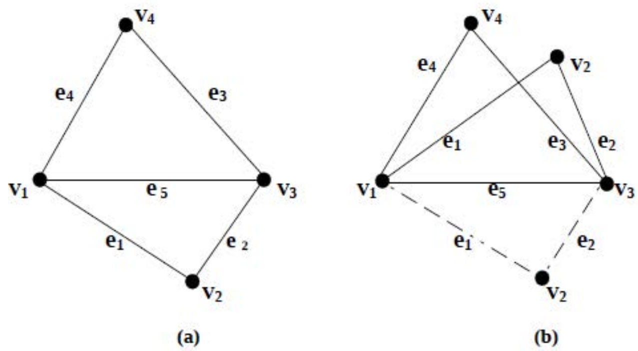


Figure 5. Equivalent frameworks which are not congruent.

Lemma 1. If $G = (V;E)$ is a graph with $|V| = \mathcal{V}$ and $p \in R^{2V}$ then $f_G^{-1}(f_G(p)) \subseteq f_{K_V}^{-1}\left(f_{K_V}(p)\right)$.

The equality holds if G is uniquely realizable up to congruence.

1.3 Motion of the Plane and a Framework

Intuitively, motion can be described as a continuous movement of the plane or of a framework. Whether a framework lying in the plane is rigid or not can be determined by investigating possible motions of it. If a motion of a frame work can be created by small perturbation to the framework keeping some portion (consisting of two or more points) fixed then the framework may be viewed as a flexible one, otherwise it is rigid. This intuitive idea is formally described below.

Definition 2. A motion f of the real plane R^2 is a function which maps $(x;t) \to f_t(x) \epsilon R^2$ at a time t for $x \in R^2$ satisfying the following conditions:

1. For each x , $f_0(x) = x$,

2. For each t and each pair of points x and y in R^2, $\| f_t(x) - f_t(y) \| = \| x - y \|$, i.e., f is an isometry (distance preserving mapping of the plane).

Thus from the definition of plane motion, we can write for a given pair of points x,y ϵ R^2,

$\| f_t(x) - f_t(y) \|^2 = \| x - y \|^2$ = constant.

If the function f is differentiable w.r.t. t then differentiating w.r.t. t we get,

$$\frac{d}{dt} \| f_t(x) - f_t(y) \|^2 = 0 \ldots \ldots \ldots (1).$$

We know for a vector α that varies on t,

$$\frac{d}{dt} \|\alpha\|^2 = \frac{d}{dt} \langle \alpha, \alpha \rangle = 2 \langle \frac{d}{dt}, \alpha \rangle.$$

This gives from Equation (1),

$$2\left\langle \frac{d}{dt}(f_t(x) - f_t(y)); f_t(x) - f_t(y)\right\rangle = 0;$$

or,

$$\langle (f'_t(x) - f'_t(y)); f_t(x) - f_t(y)\rangle = 0;$$

Similarly, motion (or, continuous deformation) of a framework $(G; p)$ in the plane is a family of functions Φ_t: $V(G) \to R^2$ of time t such that:

1. $\Phi_0(v) = p(v)$, for each v in V,
2. For each v in V, $\Phi_t(v)$ is differentiable in t (i.e., v moves along a smooth curve),
3. For each t, $\Phi_t(G)$ is a realization of G in plane,
4. The function $\Phi_t(G)$ preserves the edge lengths of G.

A plane motion f always defines a motion Φ of the framework $(G;\ \Phi_0(G))$ by restricting f to Φ over the set of points $\Phi_0(G)$. The converse is not always true.

1.4 Infinitesimal Motion of the Plane and a Framework

The movement of a framework or the plane in motion occurs under certain initial velocity map (i.e., every point x of a framework or the plane in motion is associated with a velocity vector $v(x)$). Such a velocity map is called an infinitesimal motion under certain conditions in view of small displacements of the positions. In the motion of the framework or the plane, the distance between each pair of points is presented. On the contrary, in the case of infinitesimal motion, the distance between each pair of points is preserved up to the first order derivative of the displacement. This intuition is formally described in the following definition.

Definition 4. An infinitesimal motion of the plane is a mapping $\psi : R^2 \to R^2$ such that the distance between any pair of points remain unchanged in the first order derivative with respect to t, i.e.,

$$\frac{d}{dt}\left\|(x + t\,\psi(x)) - (y + t\,\psi(y))\right\|^2 = 0$$

at $t = 0$, x, y $\in R^2$ with $x \neq y$.

If ψ is an infinitesimal plane motion then,

$$\frac{d}{dt}\left\|(x + t\,\psi(x)) - (y + t\,\psi(y))\right\|^2$$

$$= 2\,\langle \psi(x) - \psi(y), x - y + t\,\psi(x) - \psi(y)\rangle$$

At $t = 0$, $\langle \psi(x) - \psi(y), x - y\rangle = 0$. This discussion is summarized in the following lemma.

Lemma 2[7]. A vector map $\psi: R^2 \to R^2$ is an infinitesimal motion of R^2 if and only if

$$\langle \psi(x) - \psi(y), x - y\rangle = 0, \qquad \forall\ x; y \in R^2.$$

In view of Lemma 2, infinitesimal motion is an initial velocity map such that the resultant of velocity vectors at any two different points of the plane is in the perpendicular direction to the line joining the points.

Let f be a plane motion which is differentiable. Assuming $f'_t(x) = \frac{d}{dt} f_t(x)$ and substituting $t = 0$ in Equation (2) of Section 1.3 we get,

$$\langle f'_0(x) - f'_0(y); x - y\rangle = 0;$$

since $f_0(x) = x$, $f_0(y) = y$. Thus, f_0 is an orthogonal mapping. Using Lemma 2 we get the following lemma,

Lemma 3. Let f be a given plane motion which is differentiable. A velocity vector $v(x)$ associated with points in the plane such that,

$$v(x) = \frac{d}{dt} f_t(x);\ \forall x \in R^2;$$

is an infinitesimal motion of the plane.

The infinitesimal motion for a framework can be defined equivalently where the distance between each pair of adjacent vertices of the framework is preserved up to the first order changes of distances.

Definition 5. Let $G(V; E)$ be a graph and $(G; p)$ is a framework of G. An infinitesimal motion of $(G; p)$ is a map $\mu: V \to R^2$ where $\mu(v_i) = q_i$, such that at $t = 0$,

$$\frac{d}{dt}\|(p_i + tq_i) - (p_j + tq_j)\| = 0, \{\, i, j \,\} \in E.$$

Proceeding in the same way as for frameworks the following result can be obtained,

Lemma 4[15]. If μ is an infinitesimal motion of $(G; p)$ and $\mu(v_i) = q_i$, then for each edge $\{v_i, v_j\}$ of G, $\langle q_i - q_j, p_i - p_j\rangle = 0$.

2. Rigid and Flexible Frameworks

Any motion of the plane restricted to a framework in the plane gives a motion of the framework. A framework in the plane having some motion may not be extended to a motion of the plane Figure 6. A motion φ of a framework $(G; p)$ is said to be trivial[21] if it can be obtained from a motion f of the plane. A framework is called rigid if it has only trivial motions. If $(G; p)$ is not rigid it is said to be a flexible framework. Thus a flexible framework always has a motion which cannot be obtained from a plane motion.

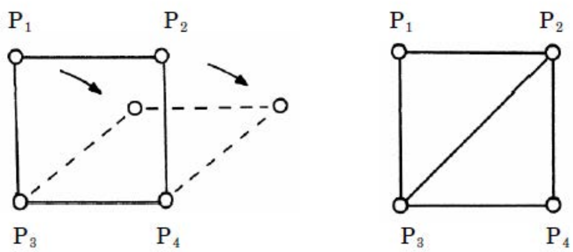


Figure 6. Square is flexible but square with a diagonal is rigid.

Example 2[21]**.** A square is flexible in R^2 since it has a nontrivial motion which deforms the square to a class of rhombus; but a square with a diagonal is rigid.

Suppose the vertices of the square framework shown in Figure 6 have initial positions at $p_1(0;1)$, $p_2(1;1), p_3(0;0), p_4(1;0)$.Vertex P_3 and P_4 are fixed in the initial positions. P_1 and P_2 can slide from their positions. Let x_1 and x_2 represent the variable positions of P_1 and P_2 as they moves along a path which preserves the edge lengths of the square at initial position. This gives,

$$\|x_1 - x_2\|^2 = 1 \text{ , } \|x_1 - p_3\|^2 = \|x_1\|^1 = 1,$$

$$\|x_2 - p_4\|^2 = \|x^2 - (1,0)\|^2 = 1.$$

Such a system of equations is called the system of edge equations for the framework. For the square framework here the above system of equations has solution set as follows,

$$x_1(t) = \left(t, \sqrt{1-t^2}\right), x_2(t) = \left(1+t, \sqrt{1-t^2}\right) \ where, t \in [0,1].$$

From this parametric solution a continuous motion $\Phi_t\colon V(G) \to R^2$ of the square framework can be defined as,

$$\varphi_t(P_1) = \left(t, \sqrt{1-t^2}\right), \quad \varphi_t(P_2) = \left(1+t, \sqrt{1-t^2}\right),$$

$$\varphi_t(P_3) = 0 \text{ , } \varphi_t(P_4) = 0 \quad \forall t \in [0,1].$$

In this parametric solution of the system of edge equations for any $t \neq 0$, the length of the diagonals of the initial square frame work are not preserved. Therefore, this cannot be a plane motion.

The example shows that rigidity or flexibility of a framework in plane depends on the nature of the solution set of the system of edge equations near the initial realization of the framework. The proposition given below describes a necessary condition for a framework to be rigid.

Proposition 1[2]**.** Let $(G;p)$ be a rigid framework of graph $G(V;E)$ in plane with QUOTE $|V| = \mathcal{V}$ then $p(V) \in R^{2V}$ must have a neighborhood $U \in R^{2V}$ such that $f_{K_V}^{-1}\left(f_{K_V}(p)\right) \cap U = f_G^{-1}(f_G(p)) \cap U$.

Proof. Let $(G;p)$ be a rigid framework. From Lemma 1 we get, $f_{K_V}^{-1}\left(f_{K_V}(p)\right) \cap U \subseteq f_G^{-1}(f_G(p)) \cap U$

for all neighbourhoods U of p . The result will be followed if we can prove

$$f_G^{-1}(f_G(p)) \cap U - f_{K_V}^{-1}\left(f_{K_V}(p)\right) \cap U = \emptyset$$

for some neighbourhood U of p. If possible let for every neighborhood U of p , $f_G^{-1}(f_G(p)) \cap U - f_{K_V}^{-1}\left(f_{K_V}(p)\right) \cap U \neq \emptyset$. Then in each neighborhood U of p there exists at least one point $p' \in f_G^{-1}(f_G(p)) \cap U - f_{K_V}^{-1}(f_{K_V}(p)) \cap U$ such that $(G;p')$ is equivalent to $(G;p)$ but not congruent. Using the algebraic approximation theory for curves[5], we get a continuous path, C , from p to p' in R^{2V} such that all points of the path $C - \{p\} \in f_G^{-1}(f_G(p)) \cap U - f_{K_V}^{-1}(f_{K_V}(p)) \cap U$, i.e., each point on $C - \{p\}$corresponds an equivalent but non-congruent framework to $(G;\ p)$. Thus along the path C , G has a non-trivial motion[6,7]. It contradicts that $(G;p)$ is rigid. Hence the result follows.

In view of Proposition 1, the rigidity of a framework $(G;p)$ may alternatively be defined as a local property of the initial realization p .

Definition 6. A framework $(G;p)$ is *rigid*[8] if there exists a real number $\varepsilon > 0$ such that for each equivalent framework $(G;q)$ of $(G;p)$ satisfying the condition $\left|\left|p(v) - q(v)\right|\right| < \varepsilon$ ($\forall v \in V$), $(G;q)$ is congruent to $(G;p)$. If $(G;p)$ is not rigid, it is called a *flexible framework*.

Let μ be an infinitesimal motion of a frame work $(G;p)$. If the plane has an infinitesimal motion ψ such that the restriction of ψ to $(G;p)$ coincides with μ then μ is called a *trivial infinitesimal motion* of $(G;p)$. $(G;p)$ is called *infinitesimally rigid* if all infinitesimal motions of the framework are trivial. A framework is called *infinitesimally flexible* if it is not infinitesimally rigid.

Proposition 2. An infinitesimally rigid framework is always rigid but the converse may not be true.

Figure 7 shows a *degenerated triangle* (A triangle is *degenerated* if it has three vertices on a line) which is rigid but not infinitesimal rigid. The rigidity of the triangle can be verified by using the definition. To prove that the triangular framework is not infinitesimally rigid let, $p_1 = (a;0)$, $p_2 = (b;0)$ and $p_3 = (0;0)$ be the locations of vertices of the triangle. If $\mu_1 = (0;0)$, $\mu_2 = (0;0)$ and $\mu_3 = (0;m)$

then

$$\frac{d}{dt}\|(p_1 + t_1) - (p_3 + t_3)\|^2 = \frac{d}{dt}\|((a,0) + t(0,0)) - ((0,0) + t(0,m))\|^2$$

$$= \frac{d}{dt}\|a, -tm\|^2 = \frac{d}{dt}(a^2 + t^2\ m^2)$$

$$= 2m^2\ t.$$

Therefore at $t = 0$,

$$\frac{d}{dt}\|(p_1 + t\mu_1) - (p_3 + t\mu_3)\|^2 = 0.$$

Similarly for the remaining pairs of vertices the derivatives vanish as above. So, $\mu = (\mu_1, \mu_2, \mu_3)$ is an infinitesimal motion of the triangle.

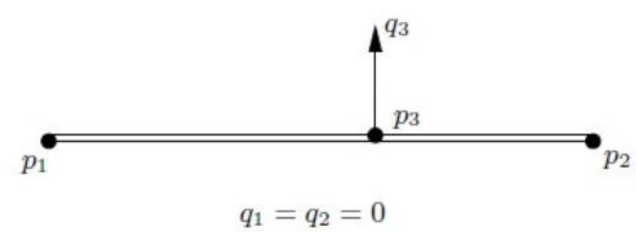


Figure 7. Rigid but not infinitesimally rigid.

If it is possible that μ is a trivial infinitesimal motion of the triangle the nit can be extended to an infinitesimal motion of the plane. We consider a point $p = (x;\ y) \neq (0;\ 0)$ in the plane. Let $\mu(p) = (u_1, u_2)$ be the infinitesimal motion at the point $(x;y)$. From Lemma 2 we get, for the points p and p_3,

$$\langle (x,y) - (0,0) | \mu(p) - \mu_3 \rangle = \langle (x,y) | (u_1, u_2 - m) \rangle = xu_1 + yu_2 - ym = 0.$$

Similarly for p and p_2, $xu_1 + yu_2 - bu_2 = 0\ a$ and for p and p_3, $xu_1 + yu_2 - au_2 = 0$. Thus for the point $p \neq (0;0)$ the infinitesimal motion $\mu(p)$ satisfies the following:

$$xu_1 + yu_2 - ym = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$xu_1 + yu_2 - bu_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$xu_1 + yu_2 - au_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The above system of equations is not consistent since $a \neq b, a \neq 0$ and $b \neq 0$. Therefore $\mu(p)$ does not exist for $p = (x;\ y) \neq (0;\ 0)$. Therefore μ cannot be a trivial infinitesimal motion of the triangle.

Hence the degenerated triangle is not infinitesimally rigid.

An alternative method for rigidity testing is calculating the rank of the rigidity matrix (discussed later).

2.1 Generic Realization of a Graph

Let $(F;+;.)$ be a field[9,10] and $V = \{v_1, v_2, \ldots, v_n\}$ be a finite subset of elements of F. Let K be a subfield of F. The set V is algebraically independent over K if there exists no polynomial $f(\neq 0)$ with coefficients from K such that $f(v_1, v_2, \ldots, v_n) = 0$. Algebraically independent is generalization of Linear independence. If $V = \{v_1, v_2, \ldots, v_n\}$ is not algebraically independent it is algebraically independent.

A framework $(G;p)$ in R^2 is said to be *generic* if the co-ordinates of all vertices are algebraically independent over the field of rational numbers. Generic realization of a graph G is a realization $(G;p)$ in which coordinates of all vertices are generic. A graph $G(V;E)$ is generically globally rigid in R^2 if all generic realizations of G are congruent[11]. From here onwards, if not mentioned otherwise, globally rigid means generically globally rigid. A rigid framework is not necessary globally rigid Figure 8.

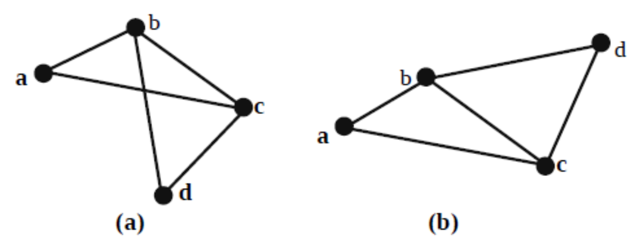


Figure 8. Both the frameworks are rigid but not globally rigid.

A framework is *degenerate* if it has three or more collinear vertices or concurrent edges. Figure 9 shows examples in which 3 or more vertices are collinear. These vertices are algebraically dependent in R^2 since each vertex coordinate is a linear combination of two remaining position vertices. Thus it is an example of degenerate framework. Existing rigidity theory is specially concerned about the generic realizations of frameworks. How to check rigidity of a degenerate framework still is an open problem.

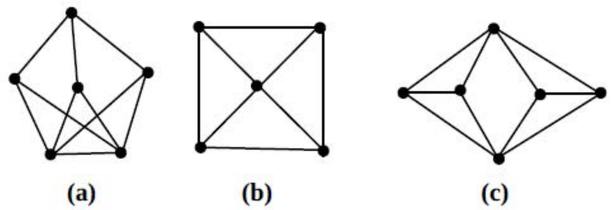


Figure 9. Non-generic graphs.

2.2 Rigidity Matrix

Let $G(V;E)$ be a graph with $|V| = v$ vertices and $|E| = e$ edges. Suppose, $f_G: R^{2v} \rightarrow R^e$ is the edge function of G, i.e., $f_G(t_1, t_2, \dots, t_v) = \left(\dots, \left\|t_i - t_j\right\|^2, \dots\right)$, where $\{i; j\} \in E$, $t_k \in R^2$, $1 \leq k \leq v$. Let $(G; p)$ be a framework of G. Starting from the initial position p of G we give a motion on $(G;p)$ such that each t_k varies with time t preserving the edge lengths, in other words, at any time t,

$$\left\|t_i(t) - t_j(t)\right\|^2 = \langle t_i(t) - t_j(t) | t_i(t) - t_j(t) \rangle = constant \dots\dots(6)$$

If t_k's are differentiable function of t, then differentiating with respect to t we get,

$$\langle t'_i(t) - t'_j(t) | t_i(t) - t_j(t) \rangle = 0$$

In view of Lemma 4, at $t = 0$, $t'_i(0)$ is the infinitesimal motion of $v_i \in V$ of $(G;p)$ where $t_i(0) = p(v_i) = p_i$. Substitution of $t'_i(0)$ by μ_i and $t_i(0)$ by p_i in equation (7) gives a set of equations each for an edge of G with μ_i as variables. At $t = 0$ Equation (7) gives a system of equations as,

$$\langle \mu_i - \mu_j | p_i - p_j \rangle = 0.$$

Solving these equations we can find the possible infinitesimal motions $\mu = (\mu_1, \mu_2, \dots, \mu_v)$ $\mu = (\mu_1, \mu_2, \dots, \mu_v)$ of $(G;p)$. The coefficient matrix of this set of equations is called the *Rigidity Matrix* of the framework[12] $(G;p)$. In R^2 the rigidity matrix has e rows and each corresponds to an edge of G. Simplifying the above equation we get,

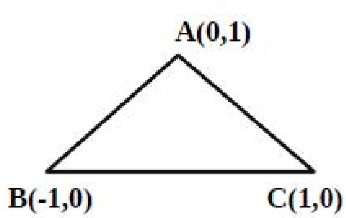
$$(p_i - p_j)\mu_i + (p_j - p_i)\mu_j = 0.$$

Since each μ_i has two components in R^2, the rigidity matrix has $2v$ columns. In order to get the rigidity matrix of the frame work $(G;p)$, instead of differentiating individual edge equations, we compute $(p_i - p_j)$ and $-(p_i - p_j)$. Let $\mu_i = (\mu_{ix}, \mu_{iy})$ and $p_i = (p_{ix}; p_{iy}) \in R^2$. $p_{ix} - p_{jx}, p_{iy} - p_{jy}$ and $p_{jx} - p_{ix}$ are the coefficients of $\mu_{ix}, \mu_{iy}, \mu_{jx}$ and μ_{jy} in the equation corresponding to the edge $\{i,j\} \in R^2$. This row has maximum four nonzero entries and others are zero. By considering the equations for all edges in similar manner, the rigidity matrix can be computed.

Example 3.[12] Let ABC be the triangle shown in Table 1 Vertices $A; B; C$ have initial positions at p1 = (0; 1); p2 = (-1,0) and p3 = (1; 0) respectively. The rigidity matrix is,

The dimension of the solution space of the system of equations given in Equation (7) can be determined by the number of independent nontrivial solutions of the system. Any framework lying in the plane always has three trivial motions and a rigid framework cannot have any nontrivial motion in the plane. Thus a framework is rigid in plane if the rigidity matrix has rank exactly equal to $2v - 3$. With the help of these properties, Hendrickson proved a result described below which is useful for testing the rigidity of a framework.

Table 1. Triangle ABC and the rigidity matrix



$$\begin{array}{c c} & \begin{array}{cccccc} p_{1x} & p_{1y} & p_{2x} & p_{2y} & p_{3x} & p_{3y} \end{array} \\ \begin{array}{c} AB \\ AC \\ BC \end{array} & \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 0 & 2 & 0 \end{bmatrix} \end{array}$$

Theorem 5[12]. A framework $(G;p)$ having v vertices is rigid in plane if and only if the rigidity matrix has rank $2v-3$.

Time complexity for calculating rank of rigidity matrix: The rigidity matrix of a framework $(G;p)$ is the coefficient matrix of a system of e equations with $2v$ variables. As the Gauss elimination method takes $O(v^3)$ time for computing the rank of this matrix therefore using Theorem 5, it is possible to test rigidity of any given framework within polynomial time.

Later Laman proposed some combinatorial properties for graphs to test the rigidity.

3. Rigidity Testing Methods

In this section we present some properties of rigid graphs in the plane and some useful methods to test the graph rigidity.

Definition 7. A graph $G(V;E)$ is rigid in the plane if each framework of G is rigid in the plane. G is redundantly rigid, if $G-e$ is rigid for each $e \in E$. G is minimally rigid if G is rigid but for any edge $e \in E$, $G-e$ is no more rigid.

The graph , $|V|>3$, is said to be an $E-Graph$ if $|E|=2|V|-3$ and for each subgraph $G'=(V',E')$ of G with $|V'|\geq 2$, $|E'|<=2|V'|-3$.

Theorem 6 (Laman[13]). A graph $G(V;E)$ is minimally rigid in R^2 if and only if $|E|=2|V|-3$ and for all subset V' of V with $|V'|>2$ the subgraph $G'=(V';E')$ has less than $2|V'|-3$ edges, i.e., a graph is an E-graph if and only if it is minimally rigid.

Since a rigid graph always has a minimally rigid subgraph, every rigid graph has an E-graph as a subgraph. Some authors refers E-graph as Laman graph. A *Laman* subgraph is a subgraph which is itself a Laman graph.

Definition 8. The edge set E of a graph $G(V;E)$ is independent in R^2 if and only if each subgraph $G'=(V';E')$ of G with $|V'|=n'$ has no more than $2n'-3$ edges. Independent subsets, E', of the edge set E are defined equivalently.

In view of Theorem 6, we have the following result.

Theorem 7. The edge set of a Laman graph is independent.

For an arbitrary independent subset of edges of a graph the following result holds.

Theorem 8. Let $G(V;E)$ be a graph. $E'\subseteq E$ is independent if rows corresponding to the edges E' in the rigidity matrix of G are independent.

Testing rigidity of graphs using Theorem 6 requires counting the number of edges in every subgraph of the graph. Since a graph with n vertices has 2^n-1 number of induced subgraphs, time complexity of testing the graph rigidity is exponential in $|V|$. In a graph G , Laman graphs can also be identified using Henneberg's construction of graphs.

3.1 Henneberg Operation

Let $G(V;E)$ be a graph. Henneberg operations on G involves the following two steps:

1. Addition of a new vertex v and two edges vu and vw to G is called a $0-extension$ operation on G Figure 10(a). The resulting graph is called the $0-extension$ of G .

2. Subdivision of an edge uv by inserting a new vertex z and adding a new edge zw for some $w\neq u$ $w\neq u$; v in G is called a $1-extension$ operation on G Figure 10(b). The resulting graph is called the $1-extension$ of G .

Definition 9. A sequence of Henneberg operations starting from K_2 to construct a new graph is known as a Henneberg construction of the graph.

Example 4. The complete bipartite graph $K(3;3)$ may be constructed from K_2 using the Henneberg operations. A 0-extension operation is applied on K_2 to form a triangle.

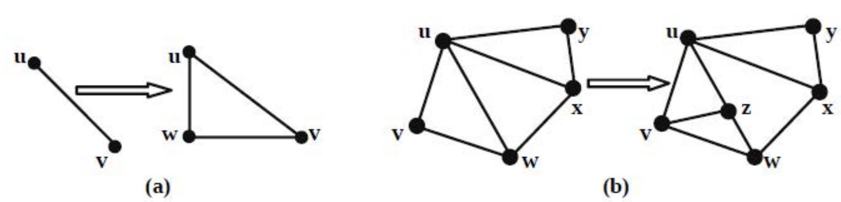


Figure 10. (a) 0-extension operation. (b) 1-extension operation.

Using 1-extension operation, each edge of the triangle is subdivided into two parts and the subdivision point is connected to the remaining vertex of the triangle Figure 11.

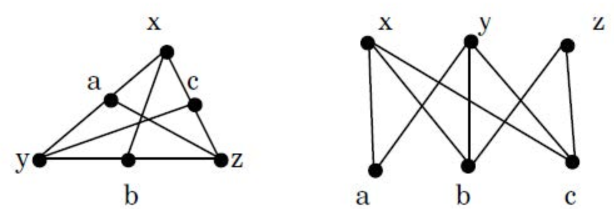


Figure 11. (a) Henneberg construction of k(3; 3). (b) Rearrangement of vertices of k(3; 3).

An extension operation is either a *0-extension* or a 1-extension operation.

Theorem 9[16]. A graph is Laman graph if and only if it can be constructed by a sequence of Henneberg operations on K_2.

Similar result holds for E -graph and minimally rigid graph, since each of these is identical to a Laman graph. An immediate consequence of the above theorem is given below.

Theorem 10[16]. Let G(V;E) be a minimally rigid graph and $G(V';E')$ is a minimally rigid subgraph of G. Then G can be obtained from G' by a sequence of Henneberg operations.

Redundant rigidity of a graph can also be verified from Henneberg's construction of the graph as stated in the theorem below.

Theorem 11[16]. G is generically redundantly rigid in R^2 if and only if G can be obtained from K_4 by a sequence of Henneberg operations and edge insertions.

3.2 Henderickson's Idea for Graph Rigidity Testing

Let $G(V;E)$ be a graph. V_1 is a set consists of two identical copies of V and $V_2 = E$. A bipartite graph $B(G)\{V_1, V_2\}$ is constructed such that each $e \in V_2$ is connected to four vertices in V_1, each of which is an end vertex of edge e in G. If $|V| = n$ and $|E| = m$ then $B(G)$ has $2n + m$ vertices and $4m$ edges. Figure 12 shows an example of such a bipartite graph for a triangle. We shortly see that the Laman's condition on the graph G can be followed based on some conditions on $B(G)$.

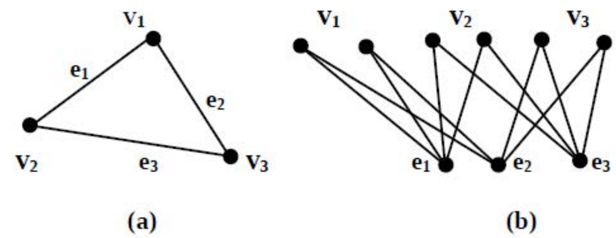


Figure 12. (a) K_3. (b) $B(G)$ for K_3.

Definition 10. A matching in G is a set of pairwise non-adjacent edges. A vertex is said to be covered by a collection of edges if the vertex appears as an end vertex of some edge in that collection. A complete matching is a matching in G which covers all vertices of the graph.

Let $G(V;E)$ be a graph. $\hat{E} \subseteq E$ is an independent subset. $\hat{V}$ is the set of vertices covered by $\hat{E}$. We consider the graph $\hat{G} = \hat{G}(\hat{V}, \hat{E})$. Suppose, $ab \in E - \hat{E}$. $\bar{G} = \hat{G} \cup \{a, b\}$. Let G' be a graph obtained from $\bar{G}$ by quadrupling an arbitrary edge of $\bar{G}$.

Theorem 12[12]. For an arbitrary edge $ab \in E - \hat{E}$, $\hat{E} \cup \{ab\}$ is an independent set in G if and only if $B(G')$ has a complete bipartite matching.

Thus for an edge $ab \in E - \hat{E}$, if $\bar{G}$ fails the matching test for every edge in $\bar{G}$ then $\hat{E} \cup \{ab\}$ cannot be independent in G. If $\hat{E} \cup \{ab\}$ is independent in G for no $ab \in E - \hat{E}$ then $\hat{G}$ has $2n' - 3$ independent edges where $\hat{G}$ has n' vertices, i.e., $\hat{G}$ is a Laman subgraph.

Lemma 13. The union of any two Laman subgraphs of G, which share common edges, is a Laman subgraph.

Lemma 13 is useful for enlarging the size of an independent subset of edges of a graph to produce a maximal independent edge set.

Theorem 14. If a maximal independent edge set of a graph $G(V;E)$ has $2|V| - 3$ edges then G is rigid.

3.2.1 Hendrickson's Algorithm

The basic idea of Hendrickson's algorithm is to find a maximal independent set of edges in a graph G using Theorem 12. Let $\hat{E}$ be an independent set of edges from E. $\hat{E}$ is called an initial basis. In each step, a new edge $ab \in E - \hat{E}$ is identified such that $\hat{E} \cup \{ab\}$ is independent. If the graph G has n vertices and an independent edge set of size $2n - 3$ is found, then the graph is a rigid graph.

1: ***procedure*** Test Rigidity ($G(V; E)$) /* u = current node */
2: ***basis*** Φ
3: ***for*** each vertex v ∈ V do
4: ***mark*** each vertex in a Laman subgraph with v and unmark all the remaining
5: ***for*** each edge {v; u} ***do***
6: ***if*** u is unmarked ***then***
7: ***if*** {v; u} is independent of basis ***then***
8: ***add*** {v; u} to basis
9: ***create*** Laman subgraph consisting of {v; u}
10: ***else*** a new Laman subgraph is identified
11: ***merge*** all Laman subgraphs with common edge
12: ***mark*** each vertex in a Laman subgraph with v
13: ***end if***
14: ***end if***
15: ***end for***
16: ***end for***
17: ***end procedure***

3.2.2 Time Complexity in Hendrickson's Algorithm

Checking whether a new edge ab is independent of the existing independent edge set $\hat{E}$ requires $O(n)$ time where n is the number of vertices of graph $G(V; E)$. Marking each vertex in a Laman subgraph and merging two Laman subgraphs needs $O(n)$ time[12]. Thus the total run-time is $O(n^2)$ which much betterthan Laman's method is.

4. Condition for Unique Realizability of Frameworks

A generic framework $(G; p)$ is uniquely realizable or generically globally rigid if all realizations equivalent to $(G; p)$ are congruent. Both redundant rigidity and vertex-connectivity have significant role for testing unique realizability of a graph.

Definition 11. A graph G is $(n+1)$-connected if it is necessary to remove at least $n+1$ vertices to increase the number of components.

Theorem 15[12]. If a generic framework $(G; p)$ is a unique realization of G in R^n then either G is a complete graph with $n+1$ vertices or

1. G is $(n+1)$-connected and
2. G is redundantly rigid in R^n.

Hendrickson conjectured that conditions the Equations (1) and (2) are sufficient for unique graph realizations. The proof for sufficiency follows immediately when $(G; p)$ lies in R, since $(G; p)$ is rigid in R if and only if it is connected. In 1991, Connelly[14] proved that this conjecture is false for $n \geq 3$. For example, in R^3 the complete Bipartite Graph $K_{5,5}$ is redundantly rigid and vertex 4-connected; though there are generic realizations where $K_{5,5}$ is not uniquely realizable[15]. Jackson and Jordan[16] established the result for $n = 2$ based on rigidity matroid.

4.1 Rigidity Matroid

An ideal or hereditary family is a collection F of subsets of a set S such that every subset of a set in F is also in F. A matroid[16] or hereditary system R_M on S is a nonempty ideal I of subsets of S with some properties. These properties are called aspects of R_M. Elements of I are called independent sets. The empty set is trivially independent as shown in Table 1.

Definition 12[16]. Let $G(V; E)$ be a graph. The $rigidity\ matroid R_M(G) = (E; I)$ is defined on the edge set E of G as $I = \{E' : E'$ is an independent set of edges in $G\}$.

Note that the characterization for independence of edge subsets are the aspects of $R_M(G)$. However, in[20] it is shown that, $R_M(G)$ is a matroid if and only if it has the following properties:

1. Φ is in $R_M(G)$.
2. $E'' \subseteq E \in R_M(G)$ then $E'' \in R_M(G)$.
3. For every $E' \subseteq E$ the maximal independent subsets of E' have the same cardinality.

Since Φ is trivially independent, property (1) is immediate. From definition of matroid we get (2). Equation (3) follows from Lemma 16.

Lemma 16[16]. Let $G(V; E)$ be a graph with $|E'| \geq 1$. If $E' \subset E$ is a maximal in dependent subset of E then $|E'| = \min \sum_{i=1}^{t} (2|X_i| - 3)$, where minimum is taken over all collection $\{X_1, X_2, \dots, X_t\}, X_i \subseteq V$ such that E' is partitioned by the edge sets of the induced subgraphs $\langle X_1\rangle, \langle X_2\rangle, \dots, \langle X_t\rangle$.

4.2 Global Rigidity Testing

Let $G(V;E)$ be a graph and $R_M(G) = (E, I)$ be the rigidity matroid. A redundantly rigid graph $G(V;E)$ is called an R_M-circuit if G has $2|V| - 2$ number of edges. Figure 13 shows some examples of R_M-circuits.

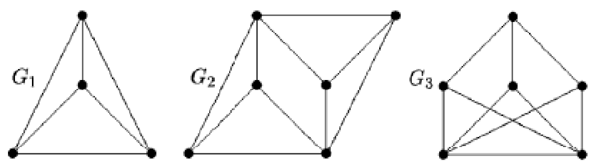


Figure 13. Examples of R_M-circuits.

Let e and f be two edges of G. e is said to be related with f if and only if either $e = f$ or both of them belong to a single R_M-circuit contained in G. This is an equivalence relation on E. Each equivalence class with respect to this relation is called an R_M-component. A graph with a single R_M-component is called R_M-connected. A vertex-3-connected graph G is called a brick if it is R_M-connected.

Theorem 17[16]. A graph is a brick if and only if it can be obtained from K_4 by edge addition and $1 - extension$ operations.

Theorem 18[16]. Any graph obtained from K_4 by sequence of edge addition and $1 - extension$ is uniquely realizable.

Theorem 19[16]. In R^2 if a generic framework $(G;p)$ is 3-connected and redundantly rigid then $(G;p)$ has a unique realization, i.e., Hendrickson's conjecture is true in R^2.

5. Rigid Graph with Ordering of Vertices

Vertex ordering of a graph is useful in testing the graph rigidity specially in distributed environments. If a graph is identified as uniquely realizable then the unique positions can be computed by existing localization techniques, e.g., semi-definite programming[17,18]. Though an arbitrary uniquely realizable graph can efficiently be recognized in a centralized environment, the realizability testing in distributed environment is still an open problem. However, some popular graphs, like bilateration graph, trilateration graph, wheel extension, triangle bar[19,20] in several real applications, can uniquely be recognized in distributed environment. In this section, the rigidity properties of these graphs are presented in the light of vertex ordering.

5.1 Bilateration Ordering

A bilateration ordering is a sequence $Y = (u_1, u_2, \dots, u_n)$ of nodes, where u_1, u_2 form a K_2 and every $\{u_i, i \geq 3\}$ is adjacent to two distinct nodes u_j and u_k for some $j, k < i$ (i.e., two nodes before u_i in Y). A graph having a bilateration ordering of nodes is called a bilateration graph Figure 14. A bilateration graph always contain $2n - 3$ edges where n is the number of vertices in it. Using Theorem 9 we reach the following result.

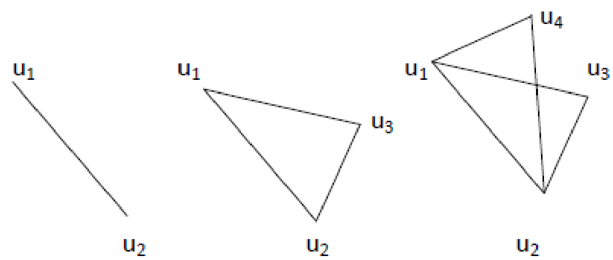


Figure 14. example of Bilateration ordering.

Theorem 20. Bilateration graphs are always minimally rigid, i.e., they are Laman graphs.

In applications, uniquely realizable graphs without having bilateration ordering are rare[21]. Recognition of uniquely realizable graphs having bilateration ordering in distributed environment is an open problem.

5.2 Trilateration Ordering

A trilateration ordering is a sequence $Y = (u_1, u_2, \dots, u_n)$ where u_1, u_2, u_3 form a triangle and every $\{u_i, i > 3\}$ is adjacent to at least three nodes u_j, u_k, u_l such that $j; k; l < i$. A trilateration graph is a graph with a trilateration ordering Figure 15. Trilateration graphs are uniquely realizable. Under the distributed environment, trilateration ordering is popularly used for location finding, though a large number of uniquely realizable graphs exist beyond trilateration graphs.

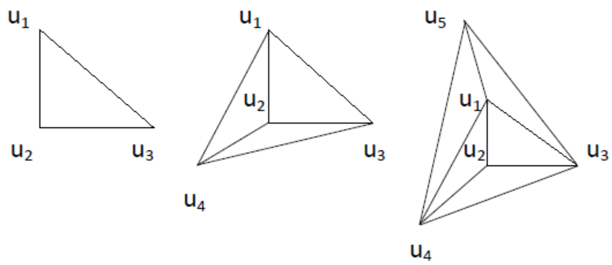


Figure 15. example of trilateration ordering.

5.3 Wheel Extension

A wheel extension is a graph having an ordering $Y = (u_1, u_2, ..., u_n)$ of nodes where u_1, u_2, u_3 form a triangle and every $\{u_i, i > 3\}$ lies in a wheel subgraph containing at least three nodes before $u_i \in Y$ Figure 16. A trilateration graph is a special case of wheel extension graph[24] which is uniquely realizable. Wheel extension graphs can efficiently be recognized in distributed environment.

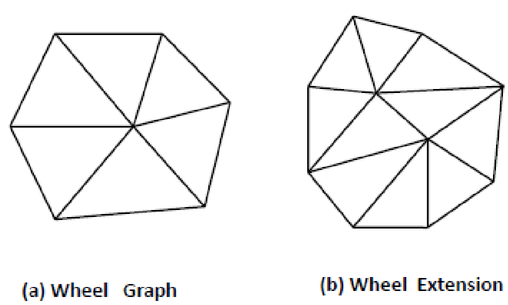


Figure 16. wheel extension graphs.

5.4 Triangle Bar

Triangle bar is a more generalized class of uniquely realizable graphs which includes trilateration graphs and wheel extension graphs as special cases. Triangle bars can efficiently be recognized under the distributed environment. We define some basic elements which are used to develop the concept of triangle bar.

5.4.1 Triangle Chain and Triangle Cycle

Let $T = (T_1; T_2; ...; T_m)$ be a sequence of distinct triangles such that for each $T_i, 2 \leq i \leq m - 1$, has two distinct edges common with T_{i-1} and T_{i+1}. Such a sequence T of triangles is called a triangle stream. Figure 17(a) shows an example of triangle stream. Let $G(T)$ be the graph-union of all T_i's in T. A node of a triangle T_i is termed a $pendant\ of\ T_i$, if the edge opposite to the vertex in T_i is shared by another triangle in T. This shared edge is called an inner side of T_i. If a graph has an unique pendent it is called a $knot$. Each triangle T_i has at least one edge which is not shared by any other triangle in T. Such a non-shared edge is called an outer side of T_i. In Figure 17(a), $T_4 = \{u, v, w\}$ has two pendants v and w, two inner sides uw and uv and one outer side vw. In a triangle stream $T = (T_1; T_2; ...; T_m)$, if T_1 and T_m has unique and distinct pendants then the triangle union $G(T)$ is termed as triangle chain. For example, 17(a) is a triangle chain. Triangle chains are rigid by construction since they involve only flips. Ina triangle chain, if T_1 and T_m share a common edge other than those shared with T_2 and T_{m-1}, then the graph union $G(T)$ is called a triangle cycle. In a triangle cycle, each triangle has exactly two inner and one outer sides. Figure 17(b) shows an example of a triangle cycle. Every wheel graph is a triangle cycle.

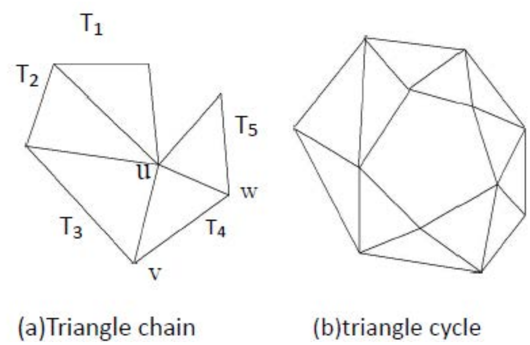


Figure 17. Triangle streams.

5.4.2 Triangle Circuit and Triangle Bridge

If $G(T)$ is neither a triangle chain nor a triangle cycle for the triangle stream T and T_1 and T_m have a unique pendant in common, then $G(T)$ is called a triangle circuit Figure 18 (c). The common pendant is called a circuit knot. For example x is the circuit knot of the triangle circuit shown in Figure 18(c).

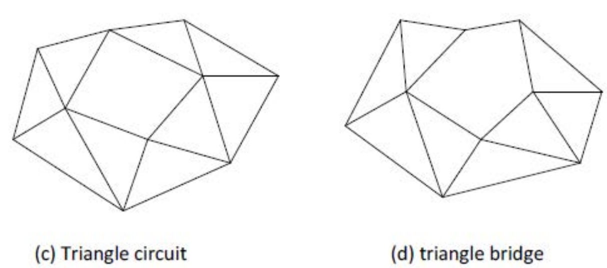


Figure 18. More examples of triangle streams.

Let $T = (T_1; T_2; \ldots; T_m)$ be a triangle stream corresponding to a triangle chain. T_1 and T_m have unique and distinct pendants. We connect these pendants by an edge e like Figure18(d). $G(T) \cup \{e\}$ is called a triangle bridge Figure 18(d).The edge e is called the bridging edge. The length of a triangle stream T is the number of triangles in it and is denoted by $l(T)$.

5.4.3 Triangle Net and Triangle Bar

Let $T = (T_1; T_2; \ldots; T_m)$ be a sequence of distinct triangles such that every T_i shares exactly one with exactly one T_j such that $1 \leq j < i$. T_1 has no pendant and T_i has exactly one pendent for $i > 1$. The graph corresponding to such a sequence of triangles is called a $triangle\ tree$. Let $G(T)$ be a triangle tree. A node v , outside $G(T)$ is called an extended node of $G(T)$ if v is adjacent to at least three nodes among which each node is either a pendant in $G(T)$ or an extended node of $G(T)$ previously added to the graph. Each of the edges which connects the extended node to a pendant or an extended knot of $G(T)$ is called an extending edge.

A graph G is called a triangle tree, if it may be generated from a triangle tree $G'(T)$ by adding one or more extended nodes and satisfying the following conditions:

1. G contains no triangle cycle, triangle circuit or triangle bridge and

2. There exists an extended node u such that every leaf knot of $G'(T)$ is connected to u by a path (extending path) containing only extending edges. The last extended node added to generate the triangle net is called an apex of the triangle net.

A graph G is called a triangle bar, if it satisfies at least one of the followings:

1. G can be obtained from a triangle cycle, triangle circuit, triangle bridge or triangle net by adding zero or more edges, but no extra node;

2. $G = B_i \cup B_j$ where B_i and B_j are triangle bars which share at least three nodes; or,

3. $G = B_i \cup \{v\}$ where B_i is a triangle bar and v is a node not in B_i, and adjacent to at least three nodes of B_i .

Triangle cycle, triangle circuit, triangle bridge and triangle nets are generically globally rigid graphs by construction. If two triangle bars B_i and B_j share three nodes in generic position, then $B_i \cup B_j$ is generically globally rigid. Leta triangle bar B be obtained from another triangle bar B' by adding a node v which is adjacent to three nodes in B0. In a generic realization of B', any node placed with three given distances from known positions has a unique location. So B is generically globally rigid.

6. Rigidity in Higher Dimensional Spaces

In the higher dimensional spaces having dimension three or more the rigidity of graphs is not much focused. However, the basic concepts of graph rigidity, namely graph realization, edge-function, equivalent frameworks, and congruent frameworks can be generalized analogously[22,23].

Motion of an $n-$ dimensional space is the collection of distance preserving some tries of the space where each point moves along a differentiable (smooth) curve to generate the is some tries at a particular time. Motion of a framework in $n-$ dimensional space can be defined in similar manner by moving the vertices of the framework. Velocity function of a motion of R^n is called the infinitesimal motion R^n and that of a framework is called the $infinitesimal\ motion$ of a framework. Trivial motions and Trivial infinitesimal motions of a framework lying in n -space are obtained from the motions and infinitesimal motions of the associated space. A framework is said to be rigid in R^n if it has only trivial motions. Likewise a framework is called infinitesimally rigid in R^n if all its infinitesimal motions are trivial infinitesimal motions. A result similar to the Theorem 1 holds for rigid frameworks in higher dimensional spaces.

Proposition 3[2]. If $(G;p)$ is a rigid framework in n -dimensional space then the realization p must have a neighborhood U in R^{nV} such that $f_G^{-1}(f_G(p)) \cap U = f_{K_V}^{-1}\left(f_{K_V}(p)\right) \cap U$ where K_V is the complete graph with V vertices.

We have already seen that for two equivalent frameworks the edge function gives equal image values, i.e., the edge lengths are preserved. Given a framework $(G;\ p)$, an equivalent framework $(G;\ q)$ satisfies the following system of equations,

$$\left|\left|x_i - x_j\right|\right|^2 = \left|\left|p_i - p_j\right|\right|^2$$

for each edge $\{v_i, v_j\}$ in G . These set of equations is called edge equations. *Inverse function theorem* suggests a technique, which identifies all possible equivalent realizations of $(G;\ p)$ in a neighborhood of $p \in R^{nV}$, to test the rigidity of the graph.

Theorem 21. *Inverse function theorem*[1]. Let $f:\ R^m \to R^m$ be a continuously differentiable function. If $x \in R^m$ is a point such that the Jacobean $J(f(x))$ is non-singular, i.e., $\det J(f(x)) \neq 0$, then there is a neighborhood V of x and a neighbourhood W of $f(x)$ such that f is one to one on V and $f:\ V \to W$ has a continuously differentiable inverse from W to V .

The following example illustrates how to use inverse function theorem for graph rigidity testing in a neighbourhood of a given framework $(G;p)$. In plane, a rigid framework may have only trivial motions. It has no motion at all if two vertices are fixed. Therefore after fixing two vertices, a motion (if any) of the graph will be a non-trivial.

Example 5[23]. Let us consider the square framework with diagonal given in Figure 19 where $p_1 = (0;1)$, $p_2 = (1;1)$, $p_3 = (0;0)$, $p_4 = (1;0)$ are the position of vertices. We fix two vertices P_3 and P_4 of the square in their initial positions. Let x_1 and x_2 represent the positions of P_1 and P_2. If the square has any motion, that will be non-trivial. If the different realizations of the square are edge distance preserving then we have,

$$\|x_1 - x_2\|^2 = 1\ ,$$

$$\|x_1 - p_3\|^2 = \|x_1\|^1 = 1, \|x_2 - p_4\|^2 = \left\|x^2 - (1{,}0)\right\|^2 = 1$$

and $\|x_2 - p_3\|^2 = 1.$

We consider a function $f:\ R^4 \to R^4$ such that,

$$f(x_1, x_2) = \left(\|x_1 - x_2\|^2, \|x_1 - p_3\|^2, \|x_2 - p_3\|^2, \|x_2 - p_4\|^2\right).$$

Being polynomial in right hand side, each component is continuously differentiable. Let $p = (p_1; p_2) = (0;1;1;1)$. Then,

$$Jf(p) = 2\begin{pmatrix} p_1 - p_2 & p_2 - p_1 \\ p_1 - p_3 & 0 \\ 0 & p_2 - p_3 \\ 0 & p_2 - p_4 \end{pmatrix} = 2\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $Jf(p)$ is non-singular, from inverse function theorem we can conclude that there is a neighborhood V of the point p and a neighborhood W of the point $f(p)$ in R^4 such that $f:\ V \to W$ is one-one. Thus it is not possible to continuously move vertices P_1 and P_2 of the square from their initial positions when P_3 and P_4 are fixed. Therefore the framework is rigid.

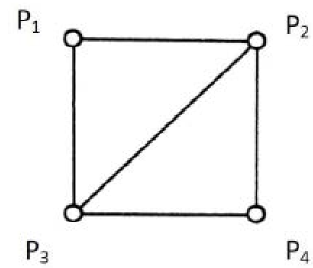


Figure 19. Square with diagonal.

On the other hand, the implicit function theorem helps to determine whether a specific realization of a framework is flexible or not.

Theorem 22. *Implicit function theorem*[1]. Let $f:\ R^{n+m} \to R^m$ be a continuously differentiable function. $p = (x;y) \in R^{n+m}$ is a point (where $x \in R^{\uparrow}n\ and\ y \in R^{\uparrow}m)$ such that $f(x,y) = 0$. If the last m columns of $Jf(p)$ are linearly independent then x has a neighborhood U in R^n such that there exists a unique continuously differentiable function $g:\ U \to R^m$ satisfying $g(x) = y$ and $(x;\ g(x)) \in f^{-1}(f(p))$ for all $x \in U$.

Using the implicit function theorem, one can find a path along which certain vertices of a framework $(G;\ p)$ can move in a neighborhood of p keeping the remaining vertices stable in their positions. The function g determines the path in such a way that $(x;\ g(x))$ remains in the solution set of the system of edge equations of $(G;p)$.

Example 6. Consider the square framework given as before let $p_1 = (0;1)$, $p_2 = (1;1)$, $p_3 = (0;0)$, $p_4 = (1;0)$ be a realization of the square and vertices P_3 and P_4 are fixed in their initial positions. x_1 and x_2 are variables positions of P_1 and P_2 in R^2. The edge equations are,

$$\|x_1 - x_2\|^2 = 1\ ,$$

$$\|x_1 - p_3\|^2 = \|x_1\|^1 = 1, \|x_2 - p_4\|^2 = \left\|x^2 - (1{,}0)\right\|^2 = 1.$$

We consider a function $f:\ R^4 \to R^3$ such that,

$$f(x_1, x_2) = \left(\|x_1 - x_2\|^2, \|x_1 - p_3\|^2, \|x_2 - p_4\|^2\right)$$

where $x_1, x_2 \in R^2$. Therefore

$$Jf(x) = 2\begin{pmatrix} x_1 - x_2 & x_2 - x_1 \\ x_1 - p_3 & 0 \\ 0 & x_2 - p_4 \end{pmatrix}.$$

At $p = (p_1, p_2) = (0,1,1,1)$

$$Jf(x) = 2\begin{pmatrix} p_1 - p_2 & p_2 - p_1 \\ p_1 - p_3 & 0 \\ 0 & px_2 - p_4 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In $Jf(p)$ the last three columns are linearly independent. Thus using implicit function theorem we conclude that, the point $0 \in R$ has a neighborhood U in R such that there exists a unique function $g: U \to R^3$ where $g(0) = (1; 1; 1)$ and $(t; g(t)) \in f^{-1}f(p)$ for all $t \in U$. Proceeding similarly as Example 2 we get, $g(t) = \left(\sqrt{(1) - t^2}, 1 + t, \sqrt{(1) - t^2}\right)$ for all $t \in U \cap [-1,1]$. Thus $(t; g(t))$ is a flexing of the square satisfying $(0; g(0)) = p$.

Definition 13. Let $(G; p)$ be a framework with v vertices, e edges and edge function $f_G: R^{nv} \to R^e$. Let $k = \max\{rank(Jf_G(x)): x \in R^{nv}\}$. A point $p \in R^{nv}$ is called a $regular\ point$ if $rank(Jf_G(p)) = k$.

Rigidity of frameworks lying in higher dimensional space can be verified using the following theorem.

Theorem 23[2]. Let $(G; p)$ be a framework with v vertices, e edges and edge function $f_G: R^{nv} \to R^e$. Let $p = (p_1, p_2, \dots, p_n) \in R^{nv}$ be a regular point of the edge function f_G and $m = dim(p)$, where $dim(p)$ is the dimension of the affinehull of $(p_1, p_2, \dots, p_n)$. Then $(G; p)$ is rigid R^n if and only if $rank(Jf_G(p)) = nv - \frac{(m+1)(2n-m)}{2}$ and $(G; p)$ is flexible in R^n if and only if $rank(Jf_G(p)) < nv - \frac{(m+1)(2n-m)}{2}$.

7. Applications

Graph rigidity has huge applications in defining formations of vehicles[24]. The unique representation of the associated framework determines the stability of the formation. The rigidity theory helps to determine the shape-variables of the appropriate potential function associated with the formation.

Another application of graph rigidity is in network localization. If the complete information of a network is available in a particular machine then the unique realizability of the network can be verified by using the results in rigidity theory. New domains from different disciplines of science and technology are regularly being added with them.

Pattern formation is an important problem in the area of multi-robot networks. In this problem, when a swarm of robots are deployed over certain area to achieve a task collectively, they may need to form a target pattern to achieve their goal. There is an important relationship between the concept of graph rigidity and pattern formation problem.

Different mathematical method and algorithms developed on the basis of combinatorial rigidity theory have several applications in protein science and mechanical engineering[25]. To study allosteric in proteins, rigidity based allosteric models and protein hinge prediction algorithms are considered as useful tools. Abridge consists of metal rods should have a rigid structure for its safety.

Another application of graph rigidity theory is in network localization problem in the domain of wireless ad-hoc networks[26] under the distributed environment. In this application, it is assumed that a sensor node is capable of measuring the distance between them and the neighbouring nodes. Localization method determines the positions of nodes satisfying the given distance measurements. If we assume the distance as the edge length then graph rigidity can be applied to find the solutions to the problem. Efficient localization in distributed setup is still an open problem which requires rigorous research attention.

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