Capacitance of Two Overlapping Conducting Spheres

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Abstract

Objectives: We calculate the capacitance of two conducting spheres, which are partially overlapping. **Methods:** Two sequences of image charges are needed to make the surfaces of the conductors equipotential by the method of images. For some special contact angles, the number of image charges is finite and they are located inside the unphysical region (that is, the conducting spheres). **Findings:** We obtain the closed-form expressions for the charges and positions of the image charges for some special contact angles from which any physical quantities including the capacitance are calculated. **Application:** The result can be applicable to estimating the capacitances of some biological cells and nanoparticles.

Keywords: Capacitance, Conducting Sphere, Method of Images, Surface Charge Density.

1. Introduction

Capacitors are one of passive elements used in electric and electronic circuits. Their capacitances depend only on the geometry of conductors and are usually calculated for parallel-plate, cylindrical, and spherical capacitors [1]. However, in dealing with parallel-plate and cylindrical capacitors with finite sizes, their edge effects are neglected. Spherical capacitor consisting of concentric conducting spheres is special in that its size is finite from the start and its exact capacitance can be easily calculated. Recently, capacitors consisting of a pair of conducting spheres whose centers do not coincide have been discussed where one surface is located inside another surface with different centers or one is located outside another [2–4].

Here, we consider the situation where two conductors with spherical surfaces S_a and S_b of radii a and b, respectively, are partially overlapping. We want to calculate the capacitance of the combined conductor using the method of images in the next section.

2. Two Overlapping Conducting Spheres

The center of one conducting sphere is chosen as the origin of the coordinate system (see Figure 1). The other conducting sphere has its center at $x = s$ on the positive x-axis and its
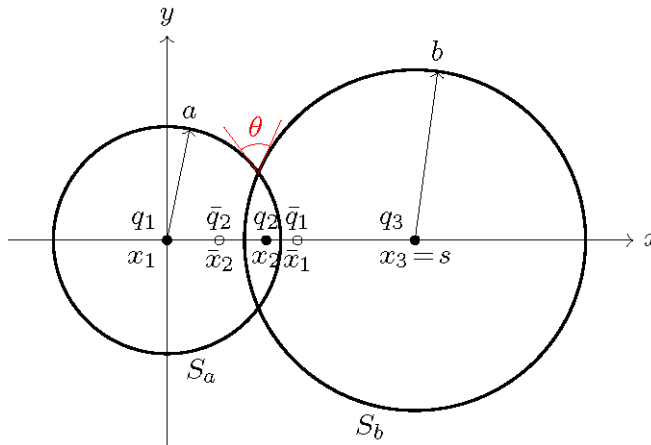
FIGURE 1. The surfaces S_a and S_b have radii a and b, respectively. The contact angle is $\theta = \pi / N$. Image charges q_{n+1} at $x = x_{n+1}$ and $\bar{q}_n$ at $x = \bar{x}_n$ are shown. Here, $b = 1.5a$ and $N = 3$ are chosen for definiteness.

surface crosses the positive x-axis at $x = s-b$. The separation between the centers of the two spheres is s with $|a - b| < s \leq a + b$.

The problem is to make their electric potential held at a constant value V. The method of images [5] is to simulate the boundary condition with suitably placed point charges with finite magnitudes inside the unphysical region surrounded by S_a and S_b. An image charge q_1 at the origin $x_1 = 0$ makes the surface S_a equipotential. Then, S_b is not equipotential any more so that an image charge $\bar{q}_1$ of q_1 with respect to S_b is required to make the potential on S_b due to charges q_1 and $\bar{q}_1$ vanishing. Now $\bar{q}_1$ is not located at the center of S_a and breaks the equipotential condition of S_a. Thus, an image charge q_2 of $\bar{q}_1$ with respect to S_a is introduced to make S_a equipotential. This process continues indefinitely in order to make the surfaces S_a and S_b each equipotential, requiring two infinite sequences $q_1, q_2, \ldots$ and $\bar{q}_1, \bar{q}_2, \ldots$ of image charges. All the image charges lie on the x-axis. In Figure 1, solid dots represent charges with the same sign as q_1, while open dots represent those with the opposite sign to q_1.

The image of the charge q_n at $x = x_n$ with respect to S_b is given by

$$\bar{q}_n = -\frac{b}{s - x_n} q_n, \tag{1}$$

$$\bar{x}_n = s - \frac{b^2}{s - x_n}. \tag{2}$$

Similarly, the image of the charge $\bar{q}_n$ at $x = \bar{x}_n$ with respect to S_a is given by

$$q_{n+1} = -\frac{a}{\bar{x}_n} \bar{q}_n, \tag{3}$$

$$x_{n+1} = \frac{a^2}{\bar{x}_n}. \tag{4}$$

From equation (1) with n replaced with $n + 1$ gives

$$\overline{q}_{n+1} = -\frac{b}{s - x_{n+1}} q_{n+1}$$

$$= \frac{ab}{(s - x_{n+1})\overline{x}_n} \overline{q}_n$$

$$= \frac{b x_{n+1}}{a(s - x_{n+1})} \overline{q}_n, \tag{5}$$

where equations (3) and (4) have been used. Equation (5) is then solved for x_{n+1}, yielding

$$\frac{s}{x_{n+1}} = 1 + \frac{b\overline{q}_n}{a\overline{q}_{n+1}}. \tag{6}$$

Eliminating $\overline{x}_n$ by combining equations (4) and (2) gives

$$\frac{a^2}{x_{n+1}} = s - \frac{b^2}{s - x_n}. \tag{7}$$

If equation (6) is substituted into equation (7), then we obtain a 2nd-order difference equation for $1/\overline{q}_n$:

$$\frac{1}{\overline{q}_{n+1}} - \frac{s^2 - a^2 - b^2}{ab} \frac{1}{\overline{q}_n} + \frac{1}{\overline{q}_{n-1}} = 0. \tag{8}$$

Since

$$-1 < \frac{s^2 - a^2 - b^2}{2ab} \leq 1, \tag{9}$$

we can define θ by

$$\cos\theta = \frac{s^2 - a^2 - b^2}{2ab} \tag{10}$$

with $0 \leq \theta < \pi$. The angle θ is shown in Figure 1. Then, equation (8) becomes

$$\frac{1}{\overline{q}_{n+1}} - e^{\pm i\theta} \frac{1}{\overline{q}_n} = e^{\mp i\theta} \left(\frac{1}{\overline{q}_n} - e^{\pm i\theta} \frac{1}{\overline{q}_{n-1}} \right) \tag{11}$$

with respective signs. Iterating $n - 1$ times,

$$\frac{1}{\overline{q}_{n+1}} - e^{\pm i\theta} \frac{1}{\overline{q}_n} = e^{\mp i(n-1)\theta} \left(\frac{1}{\overline{q}_2} - e^{\pm i\theta} \frac{1}{\overline{q}_1} \right). \tag{12}$$

From equation (1) for $n = 1$

$$\overline{q}_1 = -\frac{b}{s} q_1 \tag{13}$$

and from equation (1) for $n = 2$ combined with equations (4), (3), and (2),

$$\overline{q}_2 = -\frac{b\,q_1}{2s\cos\theta}, \tag{14}$$

so that

$$\frac{1}{\overline{q}_2} - e^{\pm i\theta}\frac{1}{\overline{q}_1} = -\frac{s}{b}\left(2\cos\theta - e^{\pm i\theta}\right)\frac{1}{q_1}$$

$$= -\frac{s}{b}e^{\mp i\theta}\frac{1}{q_1}. \tag{15}$$

Hence, we have

$$\frac{1}{\overline{q}_{n+1}} - e^{\pm i\theta}\frac{1}{\overline{q}_n} = -\frac{s}{b}e^{\mp i n\theta}\frac{1}{q_1}. \tag{16}$$

Eliminating $1/\overline{q}_{n+1}$ by subtracting the lower line from the upper line of equation (16) yields

$$\left(e^{i\theta} - e^{-i\theta}\right)\frac{1}{\overline{q}_n} = -\frac{s}{b}\left(e^{n i\theta} - e^{-n i\theta}\right)\frac{1}{q_1} \tag{17}$$

from which

$$\overline{q}_n = -\frac{b}{s}\frac{\sin\theta}{\sin n\theta}q_1. \tag{18}$$

Other quantities such as x_n, q_n, and $\overline{x}_n$ can be determined as follows. First, x_n are obtained by substituting equation (21) into equation (6) to give

$$x_n = \frac{as\sin{(n{-}1)\theta}}{a\sin{(n{-}1)\theta} + b\sin n\theta}. \tag{19}$$

Then, $\overline{x}_n$ are determined by equation (4)

$$\overline{x}_n = \frac{a\left[a\sin n\theta + b\sin{(n{+}1)\theta}\right]}{s\sin n\theta}. \tag{20}$$

From equation (1), q_n are obtained

$$q_n = \frac{b\sin\theta}{b\sin n\theta + a\sin{(n{-}1)\theta}}q_1. \tag{21}$$

When $\theta = \pi/N$ for $N = 2,3,\ldots$, we have from equation (10)

$$s = \sqrt{a^2 + b^2 + 2ab\cos(\pi/N)} \tag{22}$$

and the periodicity

$$x_{n+N} = x_n, \quad \overline{x}_{n+N} = \overline{x}_n \tag{23}$$

with

$$x_1 = 0, \ x_N = s, \ \overline{x}_N = \infty. \tag{24}$$

Hence, a finite number of image charges are used to produce the potential in the physical region outside the conductors: q_n for $n = 1, 2, \ldots, N$, and $\overline{q}_n$ for $n = 1, 2, \ldots, N-1$, which are all located in the unphysical region.

For each n the potentials on S_b due to charges q_n and $\overline{q}_n$ cancel out by construction, and similarly, the potentials on S_a due to charges q_{n+1} and $\overline{q}_n$ do. Since $q_N = (b/a)q_1$ due to equation (21), the potentials on S_a and S_b are given, respectively, by

$$V_a = \frac{1}{4\pi\epsilon_0}\frac{q_1}{a}, \ V_b = \frac{1}{4\pi\epsilon_0}\frac{q_N}{b} = V_a, \tag{25}$$

meaning that the surfaces of the conductors are equipotential. When the potential of the conducting spheres is V, we have

$$q_1 = 4\pi\epsilon_0 aV. \tag{26}$$

The electrostatic potential at any point in the physical region is given by the image charges. The surface charge density and the total charge on each surface can be calculated from the normal derivative of the potential at the surface. More simply, the total charge on S_a and S_b is, by Gauss's law, given by

$$Q = \sum_{n=1}^{N} q_n + \sum_{n=1}^{N-1} \overline{q}_n = 4\pi\epsilon_0 abV \left[\sum_{n=1}^{N} \frac{\sin\dfrac{\pi}{N}}{b\sin\dfrac{n\pi}{N} + a\sin\dfrac{(n-1)\pi}{N}} - \sum_{n=1}^{N-1} \frac{\sin\dfrac{\pi}{N}}{s\sin\dfrac{n\pi}{N}} \right]. \tag{27}$$

Hence, the capacitance of the overlapping conducting spheres is

$$C = Q/V = 4\pi\epsilon_0 ab \left[\sum_{n=1}^{N} \frac{\sin\dfrac{\pi}{N}}{b\sin\dfrac{n\pi}{N} + a\sin\dfrac{(n-1)\pi}{N}} - \sum_{n=1}^{N-1} \frac{\sin\dfrac{\pi}{N}}{s\sin\dfrac{n\pi}{N}} \right], \tag{28}$$

which is symmetric under the exchange of a and b. The capacitance C in unit of $4\pi\epsilon_0 (a+b)$ is plotted in Figure 2 as a function of the ratio b/a for three values of N.

In Figure 3, $C/\left[4\pi\epsilon_0 (a+b)\right]$ at $b/a = 1.0, 1.5, 2.0$ is depicted as a function of θ in unit of π for $N = 2, 3, \ldots, 100$. Even though the capacitance C is obtained for discrete values of N, the nature of C as a continuous function of θ is evident.
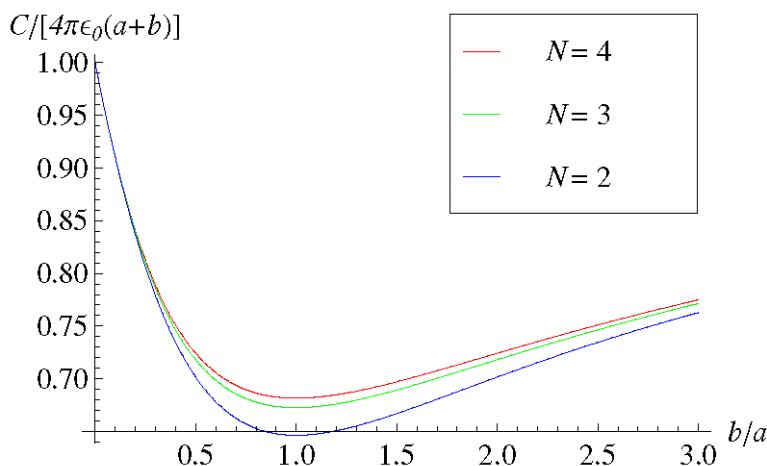
FIGURE 2. The capacitance C is plotted in unit of $4\pi\epsilon_0(a+b)$ as a function of the ratio b/a of radii of the two conducting spheres for $N=2,3,4$. For large b/a, the curves approach value 1.
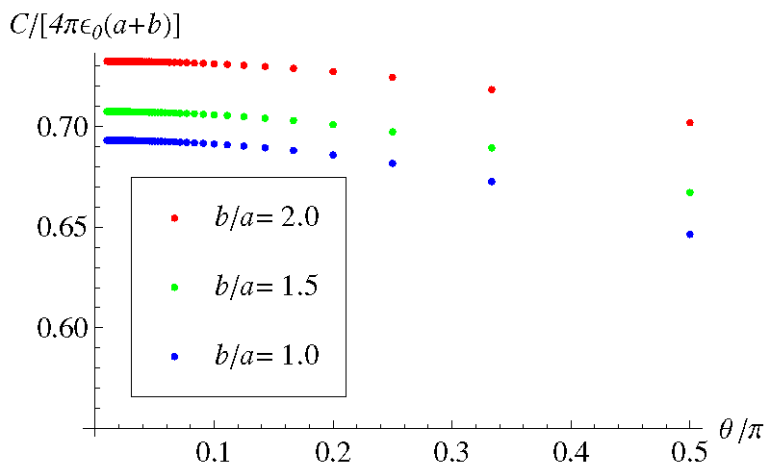

FIGURE 3. The capacitance C in unit of $4\pi\epsilon_0(a+b)$ is plotted versus the contact angle θ in unit of π for $N=2,3,\ldots,100$ at the ratios $b/a=1.0,1.5,2.0$ of radii of the two conducting spheres.

3. Concluding Remarks

Smythe [6] considered overlapping conducting spheres for the special case $N=2$. In this study, we generalized his result for any ≥ 2, where $\theta=\pi/N$ is the contact angle between the two spheres. The closed-form expressions for the charges and positions of the

image charges were obtained from which any physical quantities including the capacitance can be calculated.

Recently, electric properties of biological cells [7] and conducting nanoparticles [8] have attracted interests. The present work can provide an analytical result for the capacitances of biological cells and nanoparticles with nonspherical shapes.

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