Indian Journal of Science and Technology

Homology Group on the Dynamical Trefoil Knot

M. Abu-Saleem[1*]

[1] Department of Mathematics, Allaith University College (Girls Branch), Umm AL-Qura University, Saudi Arabia; mohammedabusaleem2005@yahoo.com

Abstract

In this article, we introduce the homology group of the dynamical trefoil knot. Also the homology group of the limit dynamical trefoil knot will be achieved. The knot group of the limit dynamical sheeted trefoil knot is presented. The dynamical trefoil knots of variation curvature and torsion of manifolds on their homology groups are deduced. Theorems governing these relations are obtained.

Keywords: Dynamical Trefoil Knot, Homolog Group, Knots.

2000 Mathematics Subject Classification: 51H20, 57N10, 57M05, 14F35, 20F34.

1. Introduction and Background

Homology theory is the axiomatic study of the intuitive geometric idea of homology of cycles on topological spaces, to any topological space X and any natural number n; one can associate a set $H_n(X)$, whose elements are called (n-dimensional) homology classes. There is a well-defined way to add and subtract homology classes, which makes $H_n(X)$ into an abelian group, called the n-th homology group of X. In heuristic terms, the size and structure of $H_n(X)$ gives information about the number of n-dimensional holes in X. For example, if X is a figure eight, then it has two holes, which in this context count as being one-dimensional. The corresponding homology group $H_1(X)$ can be identified with the group $Z \oplus Z$ of pairs of integers, with one copy of Z for each hole. While it seems very straightforward to say that X has two holes, it is surprisingly hard to formulate this in a mathematically rigorous way; this is a central purpose of homology theory.

A singular n-simplex is a continuous mapping σ_n from the standard n-simplex Δ^n to a topological space X. Notationally, one writes $\sigma_n : \Delta^n \to X$. This mapping need not be injective, and there can be non-equivalent singular simplices with the same image in X. The boundary of $\sigma_n(\Delta^n)$ denoted as $\partial_n(\sigma_n(\Delta^n))$, is defined to be the formal sum of the singular $(n-1)$-simplices represented by the restriction of σ to the faces of the standard n-simplex, with an alternating sign to take orientation into account. A formal sum is an element of the free abelian group on the simplices. Thus, if we designate the range of σ_n by its vertices $[p_0, p_1, \cdots, p_n] = [\sigma_n(e_0), \sigma_n(e_1), \ldots, \sigma_n(e_n)]$ corresponding to the vertices e_k of the standard n-simplex Δ^n (which of course does not fully specify the standard simplex image produced by σ_n, then $\partial_n(\sigma_n(\Delta^n)) = \sum_{k=0}^{n}(-1)^k[p_0, \ldots, p_{k-1}, p_{k+1}, \cdots, p_n]$ is a formal sum of the faces of the simplex image designated in a specific way. That is, a particular face has to be the image of σ_n applied to a designation of a face of Δ^n which depends on the order that its vertices are listed. Thus, for example, the boundary $\sigma[p_0, p_1]$ (a curve going from P_0 to P_1 is the formal sum or "formal difference" $[p_1]-[p_0]$. Singular chain complex, the usual construction of singular homology proceeds by defining formal sums of simplices, which may be understood to be elements of a free abelian group, and then showing that we can define a certain group, the homology group of the topological space, involving the boundary operator. Consider first the set of all possible singular n-simplices $\sigma_n(\Delta^n)$ on a topological space X.

* *Corresponding author:*
M. Abu-Saleem (mohammedabusaleem2005@yahoo.com)

This set may be used as the basis of a free abelian group, so that each $\sigma_n(\Delta^n)$ is a generator of the group. This set of generators is of course usually infinite, frequently uncountable, as there are many ways of mapping a simplex into a typical topological space. The free abelian group generated by this basis is commonly denoted as $C_n(X)$. Elements of $C_n(X)$ are called singular n-chains; they are formal sums of singular simplices with integer coefficients. In order for the theory to be placed on a firm foundation, it is commonly required that a chain be a sum of only a finite number of simplices. The boundary ∂ is readily extended to act on singular n-chains. The extension, called the boundary operator, written as $\partial_n : C_n \to C_{n-1}$ is a homomorphism of groups. The boundary operator, together with the C_n form a chain complex of abelian groups, called the singular complex. It is often denoted as $(C_\bullet(X),\partial_\bullet)$ or more simply $C_\bullet(X)$. The kernel of the boundary operator is $Z_n(X)=\ker(\partial_n)$, and is called the group of singular n-cycles. The image of the boundary operator is $B_n(X)=im(\partial_{n+1})$, and is called the group of singular n-boundaries. It can also be shown that $\partial_n\partial_{n+1}=0$. The n-th homology group of X is then defined as the factor group $H_n(X)=Z_n(X)/B_n(X)$. The elements of $H_n(X)$ are called homology classes [1-3, 5, 11–15].

Dynamical systems theory is an area of mathematics used to describe the behavior of complex dynamical systems, usually by employing differential equations or difference equations. When differential equations are employed, the theory is called continuous dynamical systems. When difference equations are employed, the theory is called discrete dynamical systems. When the time variable runs over a set which is discrete over some intervals and continuous over other intervals or is any arbitrary time-set such as a cantor set then one gets dynamic equations on time scales. Some situations may also be modeled by mixed operators such as differential-difference equations. A dynamical system in the space X is a function $q=f(p,t)$ which assigns to each point p of the space X and to each real number^t, $\infty < t < \infty$ a definite point $q \in X$ and possesses the following three properties:

a- Initial condition : $f(p,0)=p$ for any point $p \in X$.

b- Property of continuity in both arguments simultaneously :

$$\lim p \to p_0 t \to t_0 f(p,t) = f(p_0, t_0)$$

c- Group property $f(f(p,t_1),t_2)=f(p,t_1+t_2)$ [4,6-9,16].

The trefoil knot is the simplest example of a nontrivial knot. The trefoil can be obtained by joining together the two loose ends of a common overhand knot, resulting in a knotted loop. As the simplest knot, the trefoil is fundamental to the study of mathematical knot theory, which has diverse applications in topology, geometry, physics, and chemistry [10]. Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \cup Y$ obtained identifying x_0 and y_0 to a single point [13].

2. Main Result

Theorem 1. Let K be a trefoil knot then there are two types of dynamical trefoil knot $D_i : K \to \overline{K}, i=1,2$, $D_i(K) \neq K$, which induces dynamical trefoil knot $\bar{D}_i : H_n(K) \to H_n(\overline{K})$ such that $\bar{D}_i(H_n(K))$ is a free abelian group of rank ≤ 4.

Proof. Let $D_1 : K \to \overline{K}$ be a dynamical trefoil knot such that $D_1(K)$ is dynamical crossing i.e. the point of upper arc crossing touch the point of lower crossing, where $D_1(c)=p_1$ as in Figure 1(a) then we have the induced dynamical trefoil knot $\bar{D}_1 : H_1(K) \to H_1(\overline{K})$ such that $\bar{D}_1(H_1(K)) = H_1(D_1(K)) \approx H_1(S_1^1) \oplus H_1(S_2^1)$, thus $\bar{D}_1(H_1(K)) \approx Z \oplus Z$, so $\bar{D}_1(H_1(K))$ is a free abelian group of rank =2. Also, if $D_1 : K \to \overline{K}$ such that $D_1(c)=p_1$, $D_1(b)=p_2$, as in Figure 1(b) then $\bar{D}_1(H_1(K)) = H_1(D_1(K)) \approx H_1(S_1^1) \oplus H_1(S_2^1) \oplus H_1(S_3^1)$ and so $\bar{D}_1(H_1(K))$ is a free abelian group of rank =3. Moreover ,if $D_1 : K \to \overline{K}$ such that $D_1(c)=p_1$, $D_1(b)=p_2$, $D_1(a)=p_3$, as in Figure 1(c) thus $\bar{D}_1(H_1(K)) = H_1(D_1(K)) \approx H_1(S_1^1) \oplus H_1(S_2^1) \oplus H_1(S_3^1) \oplus H_1(S_4^1)$,hence $\bar{D}_1(H_1(K))$ is a free abelian group of rank $=4$. Also, we get the induced dynamical trefoil knot $\bar{D}_1 : H_0(K) \to H_0(\overline{K})$ such that $\bar{D}_1(H_0(K)) = H_0(D_1(K)) \approx Z$. Now for $n \ge 2$ we can get $\bar{D}_1$: $H_n(K) \to H_n(\overline{K})$ such that $\bar{D}_1(H_n(K)) = H_n(D_1(K)) = 0$. There is another type $D_2 : K \to \overline{K}$ such that $D_2(K)$ is dynamical trefoil knot with singularity as in Figure 1(d) then we obtain the induced dynamical trefoil knot $\bar{D}_2 : H_n(K) \to H_n(\overline{K}), n \neq 0$ such that $\bar{D}_2(H_n(K)) = H_n(D_2(K)) = 0$. Moreover, we get $\bar{D}_2 : H_0(K) \to H_0(\overline{K})$ such that $\bar{D}_2(H_0(K)) = H_0(D_2(K)) \approx Z$.

Theorem 2. The Homology group of the limit dynamical trefoil knot is a free abelian group of rank ≤1.

Proof. Let $D_1 : K \to K_1$,

$$D_2 : D_1(K) \to D_1(K_2), \ldots,$$
$$D_m : D_{m-1}(D_{m-2}) \ldots (D_1(K) \to D_{m-1}(D_{m-2}) \ldots (D_1(K_m))$$

such that $\lim\limits_{n\to\infty}(D_m(D_{m-1}) \ldots (D_1(K) \ldots))$ is a point as in Figure 2 (a,b) ,then

$H_n(\lim\limits_{m\to\infty}(D_m(D_{m-1}) \ldots (D_1(K) \ldots))) = 0,\ for\ n \ge 1$, and

$H_0(\lim\limits_{m\to\infty}(D_m(D_{m-1}) \ldots (D_1(K) \ldots))) \approx Z$. Hence, $H_n(\lim\limits_{m\to\infty}(D_m(D_{m-1}) \ldots (D_1(K) \ldots))$ is a free abelian group of rank ≤1.

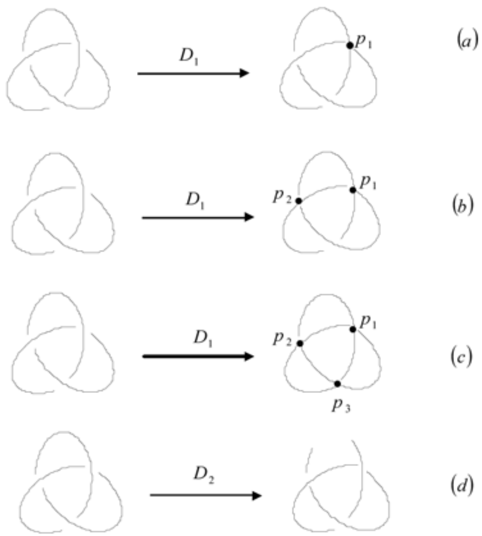

Figure 1. Dynamical trefoil knots with crossing and cutting.

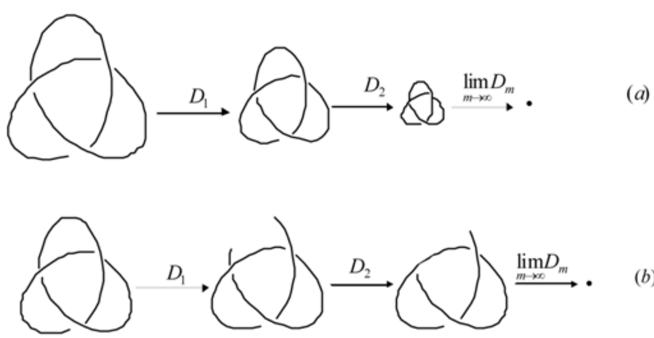

Figure 2. Limit of dynamical trefoil knot.

Theorem 3. There are different types of dynamical link graph L which represent a trefoil knot ,where $D(L) \neq L$ such that $H_n(D(L))$ is a free abelian group of rank ≤ 3.

Proof. Let L be a link graph which represent a trefoil knot and consider the following dynamical edges $D(e)=a, D(f)=c, D(g)=b$ as in Figure 3(a) then $H_1(D(L)) \approx H_1(S^1)$ and so $H_1(D(L))$ is a free abelian group of rank 1. Now, if $D(e)\neq e, D(f)\neq f, D(g)\neq g$ as in Figure 3(b) we get the same result. Also, if $D(e)=e, D(f)=f, D(g)\neq g$

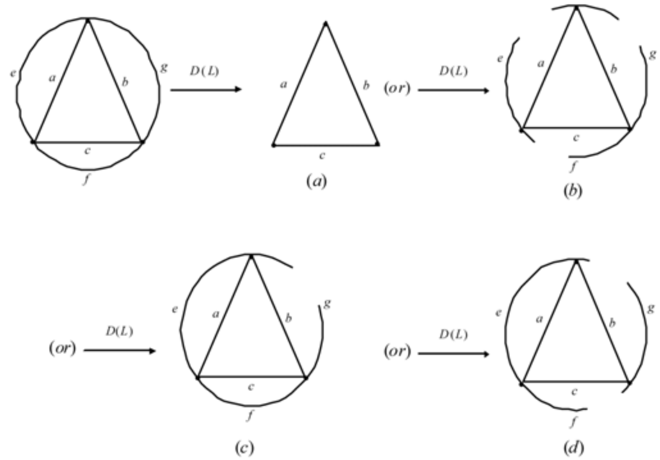

Figure 3. Dynamical link graph.

as in Figure 3(c) then, $H_1(D(L)) \approx H_1(S_1^1) \oplus H_1(S_2^1) \oplus H_1(S_3^1)$, thus $H_n(D(L))$ is a free abelian group of rank 3. Moreover, if $D(e)=e, D(f)\neq f, D(g)\neq g$ as in Figure 3(d) then $H_1(D(L)) \approx H_1(S_1^1) \oplus H_1(S_2^1)$. Hence $H_1(D(L))$ is a free abelian group of rank 2. Also for $n \neq 1$, we get $H_n(D(L))$ is a free abelian group of rank ≤ 1. Therefore $H_n(D(L))$ is a free abelian group of rank ≤ 3.

Theorem 4. Let K be the link graph of m vertices. Then $H_1(\lim_{m\to\infty}(D(K)))$ is a free abelian group of rank m, also for $n \neq 1$, $H_n(\lim_{m\to\infty}(D(K)))$ is a free abelian group of rank ≤ 1.

Proof. Let K be link graph of n vertices , then $\lim_{m\to\infty}(D(K))$ is a graph with only one vertex and m-loops as in Figure 4 , for m=3 and so $H_1(\lim_{m\to\infty}(D(K))) = H_1(\bigvee_{i=1}^{m} S_i^1) \approx \underbrace{Z \oplus Z \oplus ... \oplus Z}_{m\text{terms}}$. Hence, $H_1(\lim_{m\to\infty}(D(K)))$ is a free abelian group of rank m. Also, for $n \neq 1$, clearly we get $H_n(\lim_{m\to\infty}(D(K)))$ is a free abelian group of rank ≤ 1.

Theorem 5. Let I_1 be the closed interval $[0,1]$. Then there is a sequence of dynamical manifolds $D_i : I_i \to I_{i+1}, i=1,2,...,m$ with variation curvature and torsion such that $\lim_{m\to\infty} D_m(I_m)$ is trefoil knot and $H_n(R^3 - \lim_{m\to\infty} D_m(I_m))$ is a free abelian group of rank ≤ 1.

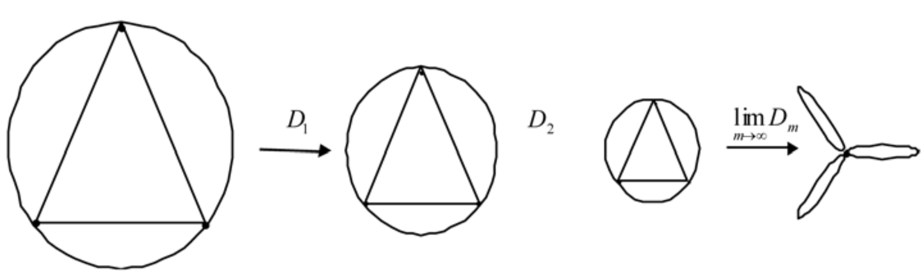

Figure 4. Limit of dynamical link graph with 3 vertices.

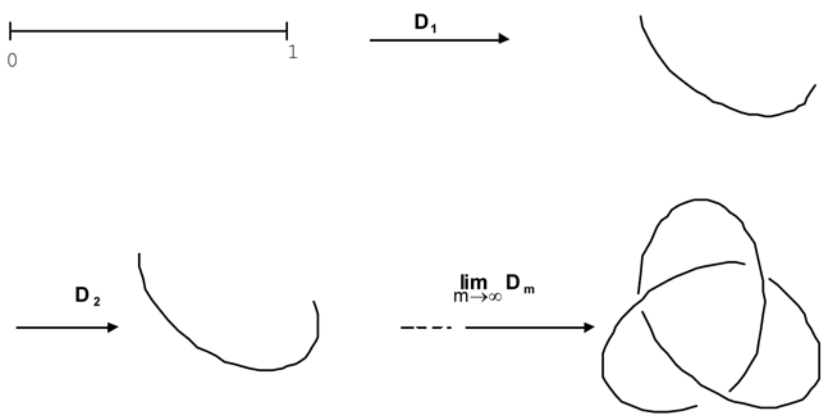

Figure 5. Limit of dynamical line segment with variation curvature and torsion.

Proof. Consider the sequence of dynamical manifolds with variation curvature and torsion: $D_1 : I_1 \to I_2, D_2 : I_2 \to I_3, ..., D_m : I_m \to I_{m+1}$ such that $\lim_{m\to\infty} D_m(I_m)$ is a trefoil knot as in Figure 5, thus, $H_1(R^3 - \lim_{m\to\infty} D_m(I_m)) \approx Z$ and $H_0(R^3 - \lim_{m\to\infty} D_m(I_m)) \approx Z$.Now for $n \geq 2$, clearly we get $H_m(R^3 - \lim_{m\to\infty} D_m(I_m)) = 0$. Therefore, $H_n(R^3 - \lim_{m\to\infty} D_m(I_m))$ is a free abelian group of rank ≤ 1.

Theorem 6. The knot group of the limit dynamical sheeted trefoil knot is a free abelian group of rank ≤ 1.

Proof. Let $\overline{K}$ be a sheet trefoil knot with boundary $\{A, B\}$ as in Figure 6 and $D : \overline{K} \to \overline{K}$ is dynamical sheeted trefoil knot of $\overline{K}$ into itself,then we get the following sequence: $D_1 : \overline{K} \to \overline{K}, D_2 : D_1(\overline{K}) \to D_1(\overline{K}), \ldots, D_m : (D_{m-1})\ldots(D_1(\overline{K})\ldots) \to (D_{m-1})\ldots(D_1(\overline{K})\ldots)$such that $\lim_{m\to\infty}(D_m(D_{m-1})\ldots(D_1(\overline{K})\ldots) = k$ where, k is a trefoil knot as in Figure 6(a) then $H_n(R^3 - k) \approx Z$ for $n = 0,1$ and $H_n(R^3 - k) = 0$, for $n \geq 2$ Also, if $\lim_{m\to\infty}(D_m(D_{m-1})\ldots(D_1(\overline{K})\ldots) =$ point as in Figure 6(b,c) then $H_n(R^3 - \lim_{m\to\infty}(D_m(D_{m-1})\ldots(D_1(\overline{K})\ldots)) = H_n(R^3 - one\ point)$. Hence $H_n(R^3 - \lim_{m\to\infty}(D_m(D_{m-1})\ldots(D_1(\overline{K})\ldots)) = 0$, for $n \neq 0$ and $H_0(R^3 - \lim_{m\to\infty}(D_m(D_{m-1})\ldots(D_1(\overline{K})\ldots))) \approx Z$. Therefore, the knot group of the limit dynamical sheeted trefoil knot is a free abelian group of rank ≤ 1.

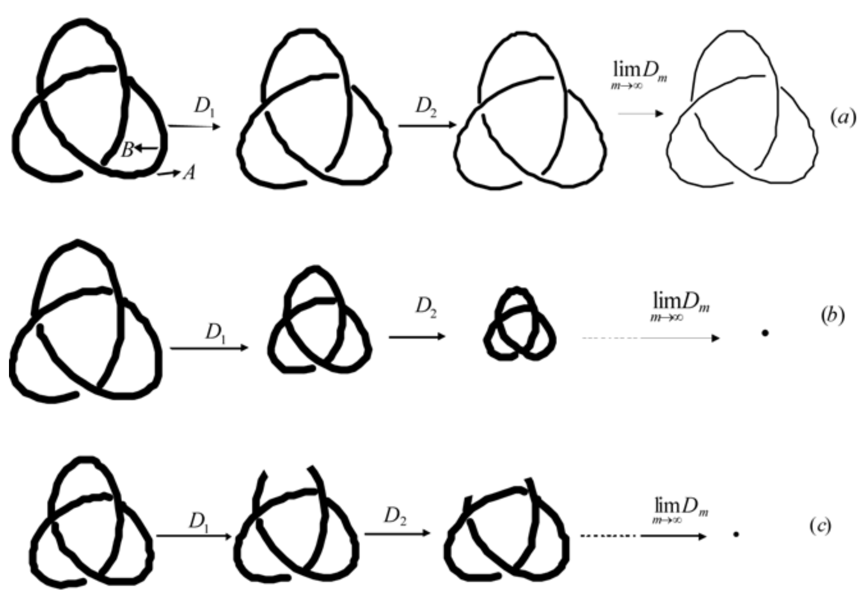

Figure 6. Limit of dynamical sheeted trefoil knot.

3. References

1. Abu-Saleem M (2008). Folding on the chaotic Cartesian product of manifolds and their fundamental group, Tamkang Journal of Mathematics, vol 39, No.4, 353–361.
2. Abu-Saleem M (2010). Folding on the wedge sum of graphs and their fundamental group, Applied Sciences, vol 12, 14–19.
3. Abu-Saleem M (2010). Homology group on manifolds and their foldings, Tamkang Journal of Mathematics, vol 41, No. 1, 31–38.
4. Abu-Saleem M (2010). Dynamical manifolds and their fundamental group, Advanced Studies in Contemporary Mathematics, vol 20, No.1, 125–131.
5. Abu-Saleem M (2010). Conditional fractional folding of a manifold and their fundamental group, Advanced Studies in Contemporary Mathematics, vol 20, No. 2, 271–277.
6. Abu-Saleem M (2011). Dynamical chaotic homotopy group and its applications, International Journal of Nonlinear Science, vol 11, No. 2, 206–212.
7. Abu-Saleem M (2011). On dynamical chaotic de sitter spaces and their deformation retracts, Proceedings of the Jangjeon Mathematical society, vol 14 , 231–238.
8. Abu-Saleem M (2012). On the dynamical hyperbolic 3-spaces and their deformation retracts. Proceedings of the Jangjeon Mathematical society 15, No. 2, 189–193.
9. Birkhoof G D (2000). Dynamical systems, Bulletin of the American Mathematics Society, vol 37, No.1, 88–121.
10. Burde G, and Zieschang H (2003). Knots,Walter de Gruyter GmbH & Co. KG.
11. Damiond G, Peltier S (2008). Computing homology generators for volumes using minimal generralized maps, Proceeding of 12 th IWCIA, volume 4959 of LNCS, USA, 63–74.
12. Giblin P J (1977). Graphs surfaces and homology:An introduction to Algebraic topology, Canada, Jon Wiley & Sons, New York.
13. Hatcher A (2002). Algebraic topology, Cambridge University Press.
14. Massey W S (1967). Algebraic topology: An introduction, Harcourt Brace and world, New York.
15. Saveliev N (2002). Invariants for homology 3-spheres, Springer -Verlage, New York.
16. Sibirsky K S (1975). Introduction to topological dynamics, Noordhoff International Pub., Leyden, The Netherlands.