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A Fixed Point Theorem for Left Amenable Semi-Topological Semigroups

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Abstract

In this note, we extend and improve the corresponding result of Takahashi[7]. Fixed point theorem for amenable semi group of non-expansive mappings.

Keywords: Amenable, Left Reversible, Non-expansive Mappings, Semi-Topological Semigroups

1. Intoduction

Let K be a subset of a Banach space E. A self mapping T on K is said to be *non-expansive* if $\|T(x)-T(y)\| \le \|x-y\|$ for all $x,y \in K$. DeMarr[3] proved the following theorem:

Theorem 1.1 For any non-empty compact convex subset K of a Banach space E, each commuting family of non-expansive self mappings on K has a common fixed point in K.

DeMarr's theorem can be further generalized for some semigroups of non-expansive self-maps on K by the following considerations.

Let S be a *semi-topological semi group*, i.e. S is a semi group with a Hausdorff topology such that for each $a \in S$, the mappings $s \to sa$ and $s \to as$ from S into S are continuous. S Is called *left reversible* if any two closed right ideals of S having non-void intersection.

Let $l^\infty(S)$ be the C*-algebra of all bounded complex-valued functions on S with supremum norm and point-wise multiplication. For each $s \in S$ and, $f \in l^\infty(S)$ denote by $l_s(f)$ and $r_s(f)$ the left and right translates of f by s respectively, that is $l_s(f)(t) = f(st)$ and $r_s(f)(t) = f(ts)$ for all $t \in S$. Let X be a closed subspace of $l^\infty(S)$ containing constants and be invariant under translations. Then a linear functional $m \in X^*$ is called a *mean* if $\|m\| = m(1) = 1$ and a *Left Invariant Mean* (LIM) if moreover $m(l_s(f)) = m(f)$ for $s \in S, f \in X$. Let $C_b(S)$ be the space of all bounded continuous complex-valued functions on S with supremum norm and $LUC(S)$ be the space of left uniformly continuous functions on S, i.e. all functions $f \in C_b(S)$ for which the mapping $s \to l_s f : S \to C_b(S)$ is continuous when $C_b(S)$ the sup-norm topology has. Then $LUC(S)$ is a C*-subalgebra of $C_b(S)$ invariant under translations and containing constant functions. S Is called *Left Amenable* if $LUC(S)$ has a LIM. The space of all right uniformly continuous functions, $RUC(S)$, and right amenability are defined similarly. The semi-topological semigroup S is called *amenable* if it is both left and right amenable, in this situation there is a mean which is both left and right invariant. Left amenable semi-topological semi groups include commutative semi groups, as well as compact and solvable groups. The free (semi) group on two or more generators is not left amenable. When S is discrete, $LUC(S) = l^\infty(S)$ and (left) amenability of S yields the (left) reversibility of S. For more details on amenability, examples and relations[1,2,4,6].

An action of S on a topological space E is a mapping $(s,x) \to s(x)$ from $S \times E$ into E such that $(st)(x) = s(t(x))$ for all $s, t \in S$, $x \in E$. The action is separately continuous if it is continuous in each variable when the other is kept fixed. Every action of S on E induces a representation of S as a semigroup of self-mappings on E denoted by S, and the two semi groups are usually identified. When the action is

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$\|s(x)-s(y)\| \le \|x-y\|$ for all $s \in S$ and $x, y \in E$.

Takahashi[7] proved a generalization of DeMarr's fixed point theorem as follows:

Theorem 1.2 Let K be a non-empty compact convex subset of a Banach space E and S be an amenable discrete semigroup which acts on K separately continuous and non-expansive. Then S has a common fixed point in K.

It is well-known that every left amenable discrete semigroup is left reversible[4], so Mitchell [5] proved the following generalization of Takahashi's theorem:

Theorem 1.3 Let K be a non-empty compact convex subset of a Banach space E and S be a left reversible discrete semigroup which acts on K separately continuous and non-expansive. Then S has a common fixed point in K.

But it is not the case that all left amenable semi-topological semi groups are left reversible as the following example shows[4]:

Example 1.4 Let S be a topological space which is regular and Hausdorff. Then $C_b(S)$ consists of constant functions only. Define on S the multiplication $st = s$ for all $s, t \in S$. Let $a \in S$ be fixed. Define $\mu(f) = f(a)$ for all $a \in S$. Then μ is a left invariant mean on $C(S)$, but S is not left reversible.

Now the question naturally arises as to whether this is true if one considers a left amenable semi-topological semi group in Takahashi's theorem instead of an amenable discrete semigroup. In this paper, we show that the answer is affirmative. Our theorem happens to be new and is not a result of any previous work.

2. Main Theorem

The space of almost periodic functions is the space of all $f \in C(S)$ such that $\{l_s f : s \in S\}$ is relatively compact in the sup-norm topology of $C(S)$ and is denoted by $AP(S)$. For any semi-topological semi group S we have the following theorem ([1], P.131 and P.164):

Theorem 2.1 (a) $f \in AP(S)$ if and only if $\{r_s f : s \in S\}$ is relatively compact in the sup-norm topology of $C(S)$.

(b) $AP(S) \subseteq LUC(S) \cap RUC(S)$.

Lemma 2.2 Let S be a semi-topological semi group which acts separately continuous and non-expansive on a compact subset M of a Banach space E. Then for each $m \in M$ and each $f \in C(M)$ we have $f_m \in LUC(S)$ where $f_m(s) = f(sm)$ $(s \in S)$.

Proof: For $f \in C(M)$ define a new function $A: M \to C(S)$ by $A(m) = f_m$, so $A(m)(s) = f(sm)$ for all $s \in S$. Put sup-norm topology on $C(S)$. We show that A is continuous. Given $m \in M$, $\varepsilon > 0$ we must find a suitable neighborhood for m such that for all m′ in it the inequality $\|A(m')-A(m)\| < \varepsilon$ holds. By continuity of f and compactness of M the function f is uniformly continuous, so there is a positive number δ such that if $u, v \in M$ and $\|u-v\| < \delta$, then $|f(u)-f(v)| < \frac{\varepsilon}{2}$. By Archimedean property of real numbers, there is a natural number k for which $\frac{1}{k} < \delta$. For each m' in the ball $B\left(m, \frac{1}{k}\right)$ and each $s \in S$ we have

$$\|sm' - sm\| < \|m' - m\| < \delta < \frac{1}{k}$$

$$|f(sm') - f(sm)| < \frac{1}{k}$$

$$B\left(m, \frac{1}{k}\right) \text{ so that if } m' \in B\left(m, \frac{1}{k}\right),$$

then

$$|f(sm') - f(sm)| = |A(m')(s) - A(m)(s)| < \frac{\varepsilon}{2}$$

For all $s \in S$. Consequently

$$\|A(m') - A(m)\| = sup\left\{|A(m')(s) - A(m)(s)| : s \in S\right\} < \varepsilon$$

Which shows that A is continuous. On the other hand for each right translate of $f_m = A(m)$ we have

$$r_a(f_m)(s) = f_m(sa) = f(sam) = f_{am}(s) = A(am)(s); \quad (s, a \in S)$$

That is $r_a f_m = A(am)$ hence $\{r_a f_m : a \in S\} = \{A(am) : a \in S\} = A(Sm)$. The set Sm is relatively compact in M and A is continuous, so $A(Sm)$ is relatively compact in the sup-norm topology of $C(S)$. Therefore by theorem

2.1 part (a) we see that $f_m = A(m) \in AP(S)$ and from part (b) $f_m \in LUC(S)$.■

Now we use the above lemma to modify Takahashi's proof[7] for left amenable semi-topological semi groups which are not necessarily discrete. Notice that in Takahashi's theorem the semi group is discrete and amenable, while in our theorem the semi group is a general semi-topological semi group which is left amenable.

Theorem 2.3 Let K be a non-empty compact convex subset of a Banach space E and S be a left amenable semi-topological semi group which acts on K separately continuous and non-expansive. Then S has a common fixed point in K.

Proof: An application of Zorn's lemma shows that there exists a minimal non-empty compact convex and S-invariant subset $X \subseteq K$. If X is a singleton we are done, otherwise apply Zorn's lemma for the second time to get a minimal non-empty compact and S-invariant subset $M \subseteq X$.

We claim that M is S -preserved, i.e. $sM = M$ for all $s \in S$ Let ν be a left invariant mean on $LUC(S)$ and define $\mu(f) = \nu(f_m)$, where f_m is defined as in lemma 2.2. By Riesz representation theorem, μ induces a regular probability measure on M (still denoted by μ) such that $\mu(sB) = \mu(B)$ for all Borel sets $B \subseteq M$ and $s \in S$. Let F be the support of μ. Each $s \in S$ defines a measurable continuous function from M into M, so by basic properties of support $F \subseteq sM$ and $\mu(sM) = \mu(M) = 1$ (see[7]). Assume that χ_F is the characteristic function of F. For each $s \in S$,

$$1 = \mu(F) = \int_M \chi_F(y)\, d\mu = \int_M \chi_F(sy)\, d\mu = \mu\left(s^{-1}F\right),$$

($s^{-1}F$ means the pre-image of F under s) again by the definition and properties of support we see that $F \subseteq s^{-1}F$, meaning that F is S-invariant. Hence $F = M$ by the minimality of M. Consequently $M = F \subseteq sM$ for each $s \in S$. But M was already S-invariant, so $sM = M$ for each s in S.

Now if M is singleton we are done, otherwise if $\delta(M) = diam(M) > 0$, we get a contradiction by DeMarr's lemma[3] which implies that

$$\exists u \in \overline{co}(M) \text{ such that } r_0 = sup\left\{\|m-u\| : m \in M\right\} < \delta(M).$$

Define, then $X_0 = \bigcap_{m \in M} B\left[m, r_0\right]$ is a non-empty (indeed $u \in X_0$) compact convex proper subset of X such that $sX_0 \subseteq X_0$ for each s in S (the inclusion follows from the fact that M is S-preserved and the action is non-expansive). But this contradicts the minimality of X. Therefore M contains only one point which is a common fixed point for the action of S. ■

Obviously every amenable discrete semigroup is a left amenable semi-topological semigroup, so we can apply theorem 2.3 to obtain the following result of Takahashi[7]:

Corollary 2.4 (Takahashi) Let K be a non-empty compact convex subset of a Banach space E and S be an amenable discrete semigroup which acts on K separately continuous and non-expansive. Then S has a common fixed point in K.

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