Indian Journal of Science and Technology, *Vol 8(12), DOI: 10.17485/ijst/2015/v8i12/71566, June 2015* *ISSN (Print) : 0974-6846*
ISSN (Online) : 0974-5645

Time Evolution of Accreting Magnetofluid Around a Compact Object-Newtonian Analysis

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Abstract

Time evolution of a magneto fluid with finite conductivity around a non-rotating compact object is presented. Along with the Maxwell equations and the Ohm law, the Newtonian limit of the relativistic fluid equations governing the motion of a finitely conducting plasma is derived. The magnetofluid is considered to possess only the colloidal components of the electromagnetic field. Moreover, the shear viscous stress is neglected, as well as the self-gravity of the disc. In order to solve the equations, we have used a self-similar solution. The main features of this solution are as follows. The azimuthal velocity is somewhat increased from the keplerian value in the equator plane to the super-keplerian values at the surface of disc. Moreover, the radial velocity is obtained proportional to the meridional velocity. Magneto fluid does not have any non-zero component of the current density. Subsequently, the electromagnetic force is vanished and does not play any role in the force balance. The pressure gradient maintains the disc structure in latitudinal direction, but this force does not have appreciable role in the radial force balance. Analogously to the parameter α in the standard model, our calculations contain one parameter η_0 which specifies the size of the electrical resistivity.

Keywords: Accretion, Accretion Disc, Magnetohydrodynamic (MHD)

1. Introduction

The main source of energy for many astrophysical objects such as Active Galactic Nuclei (AGN), binary stars and young stellar objects is plasma Processes around a central compact star. These processes include energy emission, outward angular momentum transfer, inward plasma motion and collapse on central mass. Since the process of angular momentum removal happens in slower timescales as compared to free-fall time, the infalling gas with sufficiently high angular momentum can form a disc-like structure around a central gravitating body, which can be thin or thick depending upon the geometrical shape.

About the thin accretion discs, there is a universally accepted model proposed by Shakura & Sunyaev[42] (1973, hereafter SS73). It has achieved the status of a text book paradigm, because of its relative simplicity and successful applicability. The key ingredients of this standard model are: (i) a stationary and axisymmetric assumption for the matter distribution and electromagnetic fields; (ii) a very small vertical thickness of the disc; (iii) hydrostatic equilibrium along the vertical direction; (iv) no significant pressure gradient force in the radial direction; (v) dominated rotational motion (v_r , $v_\theta \approx 0$) as Keplerian velocity profile in the azimuthal direction in the absence of self-gravity; (vi) outward angular momentum transport via turbulent shear viscosity. This viscously dissipated energy is assumed to be radiated away immediately.

For a long time, many theoretical descriptions of plasma flows were based on the approximation of the standard SS73 model Mészáros[24]; Hoshi[15] ; Meyer & Meyer-Hofmeister[23]; Wandel & Petrosian[44]; Ross et al.[36]. However, study of accretion discs beyond those simplifying assumptions of the SS73 model is of long-standing considerable theoretical interest in the context of high-energy astrophysics. For instance, including the self-gravity of the disc Fukue & Sakamoto[11]; Biermann & Duschl[6]; Rice et al[35], time dependency for the physical

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variables of the disc Mineshige & Umemura[25]; Shadmehri[38]; Khesali & Faghei[18], large vertical thickness for the disc Narayan & Yi[29]; Ghanbari et al.[13]; Samadi et al.[37], magnetic stress instead of viscous stress Kaburaki[20,21]; Shaghaghian[41] and advection (i.e. energy captures in accreting matter rather than it radiates from the disc's surface) Narayan & Yi[28]; Akizuki & Fukue[3]; Abbassi et al.[2]; Mosallanezhad et al.[27], have been remarkable improvements from the standard SS73 model.

The basic equilibrium structure of accretion discs, including thin Pringle[33]; Kaburaki[21]; Abbassi et al.[1]; Ghanbari & Shaghaghian (2009) and thick discs Arai & Hashimoto[4]; Hashimoto et al.[14]; Banerjee et al.[5]; Ghanbari & Abbassi[12] is now well developed. Nevertheless, it is not easy to follow their dynamical evolution. Mainly because the basic equations for the discs are highly nonlinear, especially when the disc is thick and there are both radial and meridional dependencies for the physical quantities. To follow the nonlinear evolution of dynamically evolving systems, in general, the technique of self-similar analyses is sometimes useful. Many researches have so far been done on self-similar disc solutions Mineshige & Umemura[25]; Mineshige et al.[26]; Ogilvie[31]; Shadmehri[39]; Khesali & Faghei[18,19]; Faghei[9]. But all of them have assumed that the disc is thin, and the accretion flow is restricted to the equatorial plane. In this case, by introducing a dimensionless self-similar variable, an initial set of nonlinear partial differential equations (i.e. Function of r and t) is reduced to a set of ordinary differential equations (i.e. Function of self-similar variable). It may be entitled deservedly one-dimensional self-similar solution. We are now concerned with time dependent behavior of a thick accretion disc in two dimensions (r and θ). Two-dimensional and time dependent solutions obviously can reveal important additional information about the dynamic of the accretion flows. Time evolution of magneto hydrodynamic equations with radial and meridional dependencies have been usually explored by numerical simulations Igumen- shchev & Abramowicz[16,17]; Proga & Begelman[34]; Ohsuga & Mineshige[30]. However, little study has been done concerning dynamical evolution of thick accretion discs in two dimensions through self-similar method. Self-similar approach allows us to investigate properties of solutions in arbitrary details, without any of the associated difficulties of simulation methods Ghanbari & Abbassi[12].

The basic equations governing the dynamics of an axisymmetric stationary magnetofluid disc around a compact object in curved space-time are given by Prasanna, Tripathy & Das[32] (1989 hereafter PTD89). The Newtonian limit of these general relativistic magnetohydrodynamic equations is carried out by Tripathy, Prasana & Das[43] (1990 hereafter TPD90) for a resistive thick disc with considering all the three components of flow velocity to be nonzero. It is thus worth investigating how this disc evolves. In the present study, we seek a two-dimensional time dependent self-similar solution for a resistive thick disc with nonzero three component fluid velocity. A brief outline of this paper is as follows: In Section 2, the general problem of constructing a model for a conducting thick disc is defined. In order to seek similarity solutions for the basic equations, a dimensionless variable is introduced and physical quantities are transformed into self-similar ones in section 3. So, in section 4, we derive transformed basic equations which are a set of coupled partial differential equations (i.e. function of self-similar variable and polar angle θ). The self-similar solutions are presented in section 5, and the effects of the input parameters are examined. Finally the conclusion is summarized in section 6.

2. Formulation

The general equations of motion for the plasma magneto fluid surrounding a compact object with mass M in a curved space-time are obtained through the energy-momentum conservation

$$T^{ij}_{;j} = 0, \tag{1}$$

Along with the Maxwell equations

$$F^{ij}_{;j} = -\frac{4\pi}{c} J^i,$$

$$F_{ij;k} = 0 \tag{2}$$

in which

$$T^{ij} = \left(\rho + \frac{P}{c^2}\right) u^i u^i - \left(\frac{P}{c^2}\right) g^{ij} - \frac{1}{4\pi c^2}\left(F^i_k F^{jk} - \frac{1}{4} g^{ij} F_{kl} F^{kl}\right),$$

is the energy-momentum tensor and F_{ij} is the electromagnetic field tensor defined as $E_\alpha = F_{\alpha t}$ and $B_\alpha = \in_{\alpha\beta\gamma} F_{\beta\gamma}$, where $\in_{\alpha\beta\gamma}$ is Levi-Civita symbol and g^{ij} is the geometric tensor. u^i that indicates the four velocity vector is related to the 3-velocity V^α through $u^a = V^{a\frac{u^0}{c}}$.

Moreover, ρ and P are the mass density and gas pressure of the fluid. J^i is the current density too and is defined through the ohm law

$$J^i = \sigma F^i_k u^k,$$

Where in σ is the electrical conductivity of the fluid. Latin indices take values 0, 1, 2, 3 (with x^0 = ct) and the Greek ones run from 1 to 3. As mentioned in introduction, we pursue the approach adopted by TPD90 for an accreting magneto fluid that is based on the following assumptions:

- The central compact star is assumed to be non rotating. Thus, the background geometry is introduced by the Schwarzschild metric

$$ds^2 = \left(1-\frac{2m}{r}\right)c^2dt^2 - \left(1-\frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2.$$

- The self-gravity of the disc is negligible in comparison with the gravitation of the central star. Therefore, the space-time structure is determined entirely by the central mass and the self-gravity is not able to disturb it.
- The disc is considered to be thick (i.e. vertical thickness H is comparable with the radius r of the disc, $H \approx r$) and axisymmetric $\left(\frac{\partial}{\partial\varphi}=0\right)$.
- The spherical polar, inertial coordinate system (r, θ, ϕ) with the origin fixed on the central object is employed and the equatorial symmetry plane placed on $\theta=\frac{\pi}{2}$
- The accreting gas is highly ionized with finite electrical conductivity, and for simplicity, σ is assumed to be constant throughout the disc.
- The effects of viscous processes are completely neglected and resistivity $\left(\eta=\frac{c^2}{4\pi\sigma}\right)$ takes the place of viscosity in the standard disc model (SS73). So, both angular momentum transfer and energy dissipation in the flow is undertaken by resistivity.
- The relativistic effects have been neglected. Therefore, the Newtonian limit (i.e. $v/c << 1$, $m << 1$) of the relativistic MHD equations (appendix A) is used. Here $m=\frac{MG}{c^2}$, G G and c are the schwarzschild mass, the gravitational constant and the light speed respectively.
- The electromagnetic field has just the poloidal component ($i.e. B_\varphi = E_\varphi = 0$).

Moreover, here, we follow a time dependent model and consider all physical quantities of the system as the functions of r, θ and t.

Applying the last foregoing assumption, the dynamical equations governing a thick accretion disc around a compact object are obtained as continuity equation

$$\rho\left[\frac{\partial V^r}{\partial r}+\frac{1}{r}\frac{\partial V^\theta}{\partial\theta}+\frac{1}{r}\left(2V^r+V^\theta cos\theta\right)\right]+V^r\frac{\partial\rho}{\partial r}+\frac{V^\theta}{r}\frac{\partial\rho}{\partial\theta}+\frac{\partial\rho}{\partial t}=0. \quad (3)$$

momentum equations

$$\rho\left[\frac{\partial V^r}{\partial t}+V^r\frac{\partial V^r}{\partial r}+\frac{V^\theta}{r}\frac{\partial V^r}{\partial\theta}+\frac{MG}{r^2}-\frac{1}{r}\left(V^{\varphi^2}+V^{\theta^2}\right)\right] + \frac{\partial P}{\partial r}+\frac{\sigma}{c^2}B_\theta\left(B_\theta V^r - B_r V^\theta\right)=0, \quad (4)$$

$$\rho\left[\frac{\partial V^\theta}{\partial t}+V^r\frac{\partial V^\theta}{\partial r}+\frac{V^\theta}{r}\frac{\partial V^\theta}{\partial\theta}+\frac{1}{r}\left(V^rV^\theta - cos\theta V^{\varphi^2}\right)\right] + \frac{1}{r}\frac{\partial P}{\partial\theta}-\frac{\sigma}{c^2}B_r\left(B_\theta V^r - B_r V^\theta\right)=0, \quad (5)$$

$$\frac{\partial V^\varphi}{\partial t}+V^r\frac{\partial V^\varphi}{\partial r}+\frac{V^\theta}{r}\frac{\partial V^\varphi}{\partial\theta}+\frac{1}{r}\left(V^rV^\varphi+V^\varphi V^\theta cos\theta\right)=0, \quad (6)$$

Maxwell equations

$$J^r = J^\theta = 0, \quad (7)$$

$$J^\varphi = -\frac{c}{4\pi r}\left[\frac{\partial}{\partial r}\left(rB_\theta\right)-\frac{\partial B_r}{\partial\theta}\right], \quad (8)$$

$$J^t = -\frac{1}{4\pi r^2}\left[\frac{\partial}{\partial r}\left(r^2 B_\theta V^\varphi\right)-\frac{r}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta B_r V^\varphi\right)\right], \quad (9)$$

$$\frac{\partial}{\partial\theta}\left(rsin\theta B_\theta\right)+\frac{\partial}{\partial r}\left(r^2 sin\theta B_r\right)=0, \quad (10)$$

$$\frac{\partial}{\partial\theta}\left(B_\theta V^\varphi\right)+\frac{\partial}{\partial r}\left(rB_r V^\varphi\right)=0, \quad (11)$$

$$\frac{\partial}{\partial t}\left(rsin\theta\, B_\theta\right)=0, \quad (12)$$

$$\frac{\partial}{\partial t}\left(r^2 sin\theta\, B_r\right)=0, \quad (13)$$

and Ohm law

$$E_r = \frac{B_\theta V^\varphi}{c}, \quad (14)$$

$$E_\theta = \frac{B_r V^\varphi}{c}, \tag{15}$$

$$J^\varphi = -\frac{\sigma}{c}\left(B_\theta V^r - B_r V^\theta\right), \tag{16}$$

$$J^t = -\frac{\sigma}{c}\left(E_\theta V^\theta - E_r V^r\right) = \frac{J^\varphi V^\varphi}{c}. \tag{17}$$

(V_r, V_θ, V_φ) are the spacial components of u^α. There are two different definitions (equations 16 and 8) for the azimuthal component of the current density. Consistency of them yields

$$J^\varphi = \frac{\partial}{\partial r}\left(rB_\theta\right) - \frac{\partial B_r}{\partial \theta} = \frac{4\pi\sigma r}{c^2}\left(B_\theta V^r - B_r V^\theta\right). \tag{18}$$

Thus, the basic equations governing the time evolution of an axisymmetric magnetofluid are represented by equations (3)-(6), (10)-(13) and (18). It is clear that the above equations are highly nonlinear that we aren't able to solve them analytically. In these circumstances, self-similar method has been very useful and widely adopted in the astrophysical literature.

3. Self-Similar Transformation

Whereas the above main equations are not easy to solve in real space, it is advantageous to transfer them to no dimensional similarity space. To this aim, it is necessary to define a dimensionless similarity variable as

$$x \equiv \frac{r}{at^n}, \tag{19}$$

Wherein α is a constant that is used to make x dimensionless. Using such a combination of radius r and instantaneous time t, all magneto hydrodynamic variables must therefore be functions of x and θ. Solutions will have the similar spacial distribution at all times, because of the self-similarity characteristics. Considering the chain rule for the transformation $(r, t) \to (x, t)$, derivatives will turn into

$$\frac{\partial}{\partial r} \to \frac{x}{r}\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - n\frac{x}{t}\frac{\partial}{\partial x}.$$

Now, we transform the physical quantities (functions of r, θ and t) into the self-similar ones (functions of x and θ)

$$V^r\left(r,\theta,t\right) = aV_r\left(x\right)V_r\left(\theta\right)t^{\in 0}, \tag{20}$$

$$V^\theta\left(r,\theta,t\right) = aV_\theta\left(x\right)V_\theta\left(\theta\right)t^{\in 0}, \tag{21}$$

$$V^\varphi\left(r,\theta,t\right) = aV_\varphi\left(x\right)V_\varphi\left(\theta\right)t^{\in 0}, \tag{22}$$

$$\rho\left(r,\theta,t\right) = R_0\rho\left(x\right)\rho\left(\theta\right)t^{\in_\rho}, \tag{23}$$

$$P\left(r,\theta,t\right) = P_0 P\left(x\right)P\left(\theta\right)t^{\in p}, \tag{24}$$

$$B_r\left(r,\theta,t\right) = \left(4\pi R_0\right)^{1\backslash 2} ab_r\left(x\right)b_r\left(\theta\right)t^{\in b}, \tag{25}$$

$$B_\theta\left(r,\theta,t\right) = \left(4\pi R_0\right)^{1\backslash 2} ab_\theta\left(x\right)b_\theta\left(\theta\right)t^{\in b}, \tag{26}$$

Also, the electrical conductivity is assumed as

$$\sigma = \sigma_0 S_0 t^{\in 0}. \tag{27}$$

Where, σ_0 and S_0 have been interpreted as a dimensional coefficient and a dimensionless free parameter, respectively. Indeed, S_0 appears in our scenario as a free parameter to play the role of magnitude of electrical conductivity. In other words, in order to investigate the effect of electrical conductivity on the physical behavior of the system, one may vary S_0. As a matter of fact, we are obligated to define σ as equation (27) so that the equation (18) is held from the dimensional consideration perspective. Substituting these expressions into the basic equations (3)–(6) and (10)-(18), we obtain the constant exponents

$$\in_0 = -\frac{1}{3}, \in_\rho =, \in_\rho - n, \in_b = \frac{1}{2}\left(\in_\rho - n\right), n = \frac{2}{3}, \tag{28}$$

and the dimensional coefficients

$$a = \left(MG\right)^{1/3}, \sigma_0 = \frac{c^2}{4\pi a^2}, P_0 = R_0 a^2. \tag{29}$$

It is worth noting here that R_0 attends in equation (23) as a dimensional coefficient with the dimension $density \times time^{-\in \rho}$. where in $\in_\rho$ is a free parameter, that the other exponents $\in_p$ and $\in_b$ are determined in terms of it. The above results indicate that each physical quantity retains a similar spacial form as the flow evolves, but the radius of the flow increases proportionally to $t^{2/3}$. The time dependent behavior of the velocities are proportional to $t^{-1/3}$, namely, they slow down as time goes by. These results are in agreement with the previous similar studies in the case of thin discs Ogilvie[31]; Khesali & Faghei[18,19]. However, the electrical conductivity scales with time as $t^{-1/3}$. This is a logical consequence from the dimensional point of view. Because, the dimension of σ_0 is $time^{-2/3}$ and it leads to this reasonable fact that the conductivity must have the dimension of $time^{-1}$.

In order to limit the allowed interval of the free parameter $\in_\rho$, the definition of mass accretion rate seems to be helpful here

$$\dot{M}(r,\theta,t) = -2\pi\int r^2 \rho V_r sin\theta d\theta - \pi\left(r^2 \rho V_\theta\right) sin\theta. \quad (30)$$

The above relation is obtained by integrating the mass conservation equation. Under transformations (19) – (23), with specified exponents (28) and Coefficients (29), we obtain

$$\dot{M}(r,\theta,t) = -2\pi MGR_0 t^{\in_\rho+1}\left\{\left[\int \rho(\theta) V_r(\theta) sin\theta d\theta\right]\right.$$
$$\left.\left[x^2\rho(x)V_r(x)\right] + \left[\frac{1}{2}x^2\rho(x)V_\theta(x)\right] \times \left[\rho(\theta)V_\theta(\theta) sin\theta\right]\right\}. \quad (31)$$

It shows that $\dot{M}$ is proportional to $t^{\in_\rho+1}$. Namely, when $\in_\rho = -1$, the mass accretion rate is independent of time and decreases in case $\in_\rho < -1$. It points to the fact that there exists an upper bound on the value of $\in_\rho$ above which the accretion rate becomes faster in process of time.

4. Transformed Basic Equations

Substituting the above transformations in the basic equations, no dimensional form of them are derived as a set of coupled partial differential equations in terms of x and θ

$$V_r(\theta)\left[\frac{dV_r(x)}{dx} + 2\frac{V_r(x)}{x} + \frac{V_r(x)}{\rho(x)}\frac{d\rho(x)}{dx}\right] + \left[-\frac{2}{3}\frac{x}{\rho(x)}\frac{d\rho(x)}{dx} + \in_\rho\right]$$
$$+\frac{V_\theta(x)}{x}\left[\frac{dV_\theta(\theta)}{d\theta} + V_\theta(\theta)cos\theta + \frac{V_\theta(\theta)}{\rho(\theta)}\frac{d\rho(\theta)}{d\theta}\right] = 0, \quad (32)$$

$$-\frac{1}{3}V_r(\theta)\left[V_r(x) + 2x\frac{dV_r(x)}{dx}\right] + V_r^2(\theta)V_r(x)\frac{dV_r(x)}{dx} + V_\theta(\theta)\frac{dV_r(\theta)}{d\theta}\frac{V_\theta(x)V_r(x)}{x}$$
$$+\frac{1}{x^2} - \frac{1}{x}\left[V_\varphi^2(x)V_\varphi^2(\theta) + V_\theta^2(x)V_\theta^2(\theta)\right] + \frac{P(\theta)}{\rho(\theta)}\frac{1}{\rho(x)}\frac{dP(x)}{dx} + S_\theta\frac{b_\theta(\theta)}{\rho(\theta)}\frac{b_\theta(x)}{\rho(x)}$$
$$\times\left[b_\theta(x)b_\theta(\theta)V_r(x)V_r(\theta) - b_r(x)b_r(\theta)V_\theta(x)V_\theta(\theta)\right] = 0, \quad (33)$$

$$-\frac{1}{3}V_\theta(\theta)\left[V_\theta(x) + 2x\frac{dV_\theta(x)}{dx}\right] + V_r(\theta)V_\theta(\theta)V_r(x)\frac{dV_\theta(x)}{dx} + \frac{1}{x}\left[V_\theta(\theta)\frac{dV_\theta(\theta)}{d\theta}V_\theta^2(x)\right.$$
$$\left. + V_r(\theta)V_\theta(\theta)V_r(x)V_\theta(x) - V_\varphi^2(\theta)cos\theta V_\theta^2(x)\right] + \frac{1}{\rho(\theta)}\frac{dP(\theta)}{d\theta}\frac{1}{x}\frac{P(x)}{\rho(x)}$$
$$-S_0\frac{b_r(\theta)}{\rho(\theta)}\frac{b_r(x)}{\rho(x)}\left[b_\theta(x)b_\theta(\theta)V_r(x)V_r(\theta) - b_r(x)b_r(\theta)V_\theta(x)V_\theta(\theta)\right] = 0, \quad (34)$$

$$-\frac{1}{3}V_\varphi(\theta)\left[V_\varphi(x) + 2x\frac{dV_\varphi(x)}{dx}\right] + V_r(\theta)V_\varphi(\theta)V_r(x)\left[\frac{dV_\varphi(x)}{dx} + \frac{V_\varphi(x)}{x}\right]$$
$$+V_\theta(\theta)\left[\frac{dV_\varphi(\theta)}{d\theta} + cos\theta V_\varphi(\theta)\right]\frac{V_\varphi(x)V_\theta(x)}{x} = 0, \quad (35)$$

$$\frac{1}{sin\theta b_r(\theta)}\frac{d}{d\theta}\left[sin\theta b_\theta(\theta)\right]=-\frac{1}{xb_\theta(x)}\frac{d}{dx}\left[x^2b_r(x)\right], \tag{36}$$

$$\frac{1}{b_r(\theta)V_\varphi(\theta)}\frac{d}{d\theta}\left[V_\varphi(\theta)b_\theta(\theta)\right]=\frac{1}{b_\theta(x)V_\varphi(x)}\frac{d}{dx}\left[V_\varphi(x)xb_r(x)\right], \tag{37}$$

$$\frac{1}{b_\theta(x)}\frac{d}{dx}\left[xb_\theta(x)\right]=1+\frac{3}{2}\in_b, \tag{38}$$

$$\frac{1}{xb_r(x)}\frac{d}{dx}\left[x^2b_r(x)\right]=2+\frac{3}{2}\in_b. \tag{39}$$

$$\begin{aligned}J^\varphi &= S_0\left[b_\theta(x)b_\theta(\theta)V_r(x)V_r(\theta)-b_r(x)b_r(\theta)V_\theta(x)V_\theta(\theta)\right]\\ &=\frac{1}{x}\left\{b_\theta(\theta)\frac{d}{dx}\left[xb_\theta(x)\right]-b_r(x)\frac{db_r(\theta)}{d\theta}\right\},\end{aligned} \tag{40}$$

One admissible solution set for the nondimensional part of the magnetic field that satisfies the equation (36) is given by

$$b_r(x)b_r(\theta)\equiv|-x^{k-1}sin^{k-1}\theta cos\theta, \tag{41}$$

$$b_\theta(x)b_\theta(\theta)\equiv x^{k-1}sin^k\theta, \tag{42}$$

where k is a constant that can be determined from the equations (38) and (39) as

$$k=\frac{1}{2}\left(1+\frac{3}{2}\in_\rho\right). \tag{43}$$

k is negative for $\in_\rho < -1$. Now, using the equations (25) and (26), we recover the magnetic field components from the self-similar forms as

$$B_r(r,\theta,t)=-(4\pi R_0)^{\frac{1}{2}}(MG)^{\frac{1}{3}(2-k)}r^{k-1}sin^{k-1}\theta cos\theta, \tag{44}$$

$$B_\theta(r,\theta,t)=(4\pi R_0)^{\frac{1}{2}}(MG)^{\frac{1}{3}(2-k)}r^{k-1}sin^k\theta. \tag{45}$$

It is seen that the time does not appear in the disc's poloidal magnetic field explicitly. As a result, it has just the same configuration as the stationary disc (TPD90), which represents constant field lines parallel to the meridional plane. Substituting equations (41) and (42) in consistency equation (40), it simplifies as

$$S_0\left[V_r(x)V_r(\theta)+cos\theta V_\theta(x)V_\theta(\theta)\right]=\frac{k-1}{xsin^2\theta}. \tag{46}$$

Thus, the remaining unused equations are represented by equations (32)–(35), (37) and (46). While the undetermined physical variables are both x and θ dependencies of V^r, V^θ, V^ϕ, ρ and P.

5. Self-Similar Solutions

We can first obtain azimuthal velocity by equation (37) which its left side is just a function of θ and its right side is a function of x. Thus, both sides must be equal to a constant α. Using the equations (41) and (42), the ordinary differential equation (37) can be solved simply by the separating variables method and the azimuthal velocity is obtained

$$V_\varphi(x)V_\varphi(\theta)=L(xsin\theta)^l, \tag{47}$$

where $l = \alpha - k$ and L is an integration constant. For $l=-\frac{1}{2}$ first two terms of equation (35) vanish each other and the equation reduces to

$$V_r(x)V_r(\theta)+cos\theta V_\theta(x)V_\theta(\theta)=0. \tag{48}$$

This equation along with the equation (46) set a lower limit on the value of conductivity non-dimensional coefficient as

$$S_0 \gg \frac{1-k}{x sin^2\theta}. \tag{49}$$

In fact, equation (48) causes azimuthal current to be zero as well as poloidal current. Also, by means of equation (22), the azimuthal velocity is attained in real space

$$V^{\varphi}(r,\theta,t) = L\sqrt{\frac{MG}{r sin\theta}} = \frac{L}{\sqrt{sin\theta}} u_k(r). \tag{50}$$

We can claim that our disc has quasi-keplerian rotation, while $u_k(r) = \sqrt{\frac{MG}{r}}$ is defined as Keplerian velocity in the standard SS73 model. Keplerian rotation requires the equality of centrifugal and gravitation terms in the radial component of momentum equation (4). Therefore, other forces (i. e., magnetic force, the pressure gradiant,...) do not play any appreciable role in the radial force balance. This point should be checked when equations have been solved completely. It seems advantageous here to compare our time dependent solution (equation 50) with its peer for a stationary disc (TPD90)

$$V^{\varphi} \; \alpha \; \frac{1}{r sin\theta},$$

It declares that the disc rotates sub-Keplerian in the case of steady state. Now, we return to the unused remaining equations (32), (33) and (34). By employing equation (48), the continuity equation (32) is simplified as

$$\begin{aligned} -cot\theta V_{\theta}(\theta)\left[\frac{dV_{\theta}(x)}{dx} + \frac{V_{\theta}(x)}{x} + \frac{V_{\theta}(x)}{\rho(x)}\frac{d\rho(x)}{dx}\right] \\ + \frac{V_{\theta}(x)}{x}\left[\frac{dV_{\theta}(\theta)}{d\theta} + \frac{V_{\theta}(\theta)}{\rho(\theta)}\frac{d\rho(\theta)}{d\theta}\right] \\ + \left[\epsilon_{\rho} - \frac{2}{3}\frac{x}{\rho(x)}\frac{d\rho(x)}{dx}\right] = 0, \end{aligned} \tag{51}$$

Multiplying equation (34) by $cot\theta$ and adding equation (33), and using relations (41), (42), (47) and (48), the following simple equation is given

$$\begin{aligned} \frac{1}{\rho(x)\rho(\theta)}\left[P(\theta)\frac{dP(x)}{dx} + \frac{P(x)}{x}cot\theta\frac{dP(\theta)}{d\theta}\right] \\ - \frac{L^2}{x^2 sin^3\theta} + \frac{1}{x^2} = 0. \end{aligned} \tag{52}$$

In the above equation, which can be regarded as the equation of force balance, the terms in bracket are the pressure gradient and the next terms are centrifugal and gravitation, respectively. It is imperative here to mention that due to the condition (48), the magnetic and inertia terms have been vanished. Moreover, the equation of force balance demonstrates that on the equator $\theta = \pi/2$ and for $L = 1$ the pressure gradient $\frac{dP}{dx}$ becomes zero. So, equality of gravity and centrifugal forces will be hold which is essential for Keplerian rotation. It should be also mentioned that according to the equation (48), at the equator, disc has not accretion on the central object ($V_r = 0$).

Now, we encounter two set of coupled non-linear partial differential equations that the solution of them, in general, is a so difficult task. Therefore, in order to simplify the equations, we presume that density follows a power law of the rescaled radial distance as

$$\rho(x) = x^{\frac{3}{2}\epsilon_{\rho}}. \tag{53}$$

Subsequently, the third bracket of equation (51) has been omitted and this equation reduce as

$$\begin{aligned} -cot\theta V_{\theta}(\theta)\left[\frac{dV_{\theta}(x)}{dx} + \frac{V_{\theta}(x)}{x} + \frac{V_{\theta}(x)}{\rho(x)}\frac{d\rho(x)}{dx}\right] \\ + \frac{V_{\theta}(x)}{x}\left[\frac{dV_{\theta}(\theta)}{d\theta} + \frac{V_{\theta}(\theta)}{\rho(\theta)}\frac{d\rho(\theta)}{d\theta}\right] = 0. \end{aligned} \tag{54}$$

We assume that disc is extend to about $\pi/3$ on either side of equatorial plan and the structure of the disc is symmetric to this plane ($V_{\theta}(\theta = \pi/2) = 0$). Applying this assumption on equation (54), we obtain

$$\frac{dV_{\theta}}{d\theta} = 0, \quad \theta = \frac{\pi}{2}$$

For other angles ($\theta \neq \pi/2$), there are two equations and three variables to be determined, namely P,ρ and V^{θ}, one needs an extra equation for closing the system. Because the flow is able to radiate efficiency, we can substitute the polytrophic equation instead of the energy equation as a relation between pressure and density

$$P = K\rho^{\gamma}, \tag{55}$$

Where γ is the adiabatic index and K is a constant. Also, in vertical direction, the adiabatic assumption is a simple way to solve the equations, and then enables us to calculate the dynamical quantities (see e.g., TPD90). If we come back to self-similar transformations and exert equation (55), the constant exponents and the dimensional coefficients for density, pressure and magnetic field are obtained as

$$\in_\rho = -\frac{2}{3}\frac{1}{\gamma-1}, \in_P = -\frac{2}{3}\frac{\gamma}{\gamma-1}, \in_b = -\frac{1}{3}\frac{\gamma}{\gamma-1},$$

$$P_0 = KR_0^\gamma = R_0 a^2. \quad (56)$$

The above results reveal that the time dependence of density, the pressure and the poloidal magnetic field vary with γ, in the manner that they fall off in time for $\gamma > 1$. Subsequently, the mass accretion rate is proportional to $t^{\frac{\gamma-5/3}{\gamma-1}}$. When $\gamma = 5/3$, the mass accretion rate is independent of time and decreases in $1 < \gamma < 5/3$. Also, k parameter which specify the spacial behavior of the magnetic field, equation (43), take the form

$$k = \frac{1}{2}\frac{\gamma-2}{\gamma-1}$$

It shows that the spacial behavior of the magnetic field depends on γ, as well as its temporal behavior. Now, equations (52) and (54) are rewritten as

$$\gamma\rho(\theta)^{\gamma-2}\left[\frac{3}{2}\in_\upsilon \rho(\theta) + cot\theta\frac{d\rho}{d\theta}\right] - \frac{L^2}{sin^3\theta} + 1 = 0. \quad (57)$$

$$\frac{x}{V_\theta(x)}\left[\frac{dV_\theta(x)}{dx} + \frac{V_\theta(x)}{x} + \frac{3}{2}\in_\rho \frac{V_\theta(x)}{x}\right] = \frac{1}{cot\theta V_\theta(\theta)}\left[\frac{dV_\theta(\theta)}{d\theta} + \frac{V_\theta(\theta)}{\rho(\theta)}\frac{d\rho(\theta)}{d\theta}\right], \quad (58)$$

In deriving the above equations, equation (53) has been used. The left hand side of equation (58) is a function of only x, and its right hand side is a function of only θ. Since this equation must be satisfied for all values of x and θ, one must conclude that each side of equation must be equal to the same constant value. We will choose the constant to be ξ. Integrating them yields

$$V_\theta(x) = x^\beta, \quad V_\theta(\theta)\rho(\theta) = sin^\xi\theta, \quad (59)$$

Where $\beta = \xi + \frac{3}{2}\in_\rho - 1$. On the other hand, the equation (57) can be numerically solved with a proper boundary condition. By using the equation (59), we can rewrite equation (57) with respect to function of V_θ. Thus, we have

$$cot\theta\frac{dV_\theta}{d\theta} + \left(\frac{1}{\gamma-1} - \xi cot^2\theta\right)V_\theta(\theta) + \left(\frac{L^2}{sin^3\theta} - 1\right)\frac{sin^{-\xi(\gamma-1)}\theta}{\gamma}V_\theta(\theta)^\gamma = 0. \;\; \pi/6 \le \theta \le \pi/2 \quad (60)$$

Now, by choosing boundary condition as V_θ (θ = 0.49π) = 0.0001, this equation can be solved to conclude meridional behavior of the velocities and density. By using the analytical solutions achieved, we can derive the distribution of poloidal components of velocity and density in real space. With the assistance of the equations (20), (21) and (23), these variables are transformed as

$$\rho(r,\theta,t) = R_0\left(\sqrt{MG}\, r_d^{-3/2}\right)^{\frac{2}{3}n_\rho}\left(\frac{r}{rd}\right)^{-n_\rho}\frac{sin^\xi\theta}{V_\theta(\theta)} = \rho_0\left(\frac{r}{r_d}\right)^{-n_\rho}\frac{sin^\xi\theta}{V_\theta(\theta)},$$

$$V^r(r,\theta,t) = -\left[(MG)^{\frac{1-\beta}{3}} r_d^\beta t_0^{-\frac{2\beta+1}{3}}\right]\left(\frac{r}{r_d}\right)^\beta\left(\frac{t}{t_0}\right)^{-\frac{2\beta+1}{3}} cot\theta V_\theta(\theta) = -V_0\left(\frac{r}{r_d}\right)^\beta\left(\frac{t}{t_0}\right)^{-\frac{2\beta+1}{3}} cot\theta V_\theta(\theta),$$

$$V^\theta(r,\theta,t) = \left[(MG)^{\frac{1-\beta}{3}} r_d^\beta t_0^{-\frac{2\beta+1}{3}}\right]\left(\frac{r}{r_d}\right)^\beta\left(\frac{t}{t_0}\right)^{-\frac{2\beta+1}{3}} V_\theta(\theta) = V_0\left(\frac{r}{r_d}\right)^\beta\left(\frac{t}{t_0}\right)^{-\frac{2\beta+1}{3}} V_\theta(\theta),$$

wherein r_d is the disc radius and $t_0 = \frac{1}{\Omega_k} = \left(\frac{r_d^3}{MG}\right)^{\frac{1}{2}}$ is defined as dynamical timescale that is the shortest timescale of the disc. The typical timescales in which the disc structure may vary should be larger than t_0. In addition, inserting equations (48) and (59) into the equation (31), mass accretion rate can be written as

$$\dot{M}(r,\theta,t) = -2\pi MGR_0 t^{\epsilon_\rho+1} x^{\xi+1}\left\{-\int \cot\theta \sin^{\xi+1}\theta d\theta + \frac{1}{2}\sin^{\xi+1}\theta\right\}$$

$$= \dot{M}_0 r^{\xi+1}\sin^{\xi+1}\theta = \left\{-\frac{1}{\xi+1}+\frac{1}{2}\right\} t^{\epsilon_\rho-\frac{2}{3}\left(\xi-\frac{1}{2}\right)}, \quad (61)$$

Where $\dot{M}_0 = 2\pi(MG)^{\frac{2-\xi}{3}} R_0$. Based on the above equation, $\dot{M}$ is a function of r and θ unless $\xi = -1$ and $\xi = 1$. For $\xi = -1$, mass accretion rate gets infinite value and for $\xi = 1$ it becomes zero. Because of this, there exists a lower as well as an upper bound for parameter ξ. In Figures 1–3, we show the θ-direction of distribution of physical quantities with different sets of input parameters L, γ, ξ. As we expect, the density distribution $\rho(\theta)$ and pressure $P(\theta)$ will grow as θ increases, and near the equator will get maximum, as shown in the Figures. On the contrary, the radial and meridional components of velocity (i. e., $V_r(\theta)$ and $V_\theta(\theta)$) from the value of zero at the equator will increase toward the surface. The effect of angular momentum parameter L is given in Figure 1. It should be noted that L parameter only affects meridional structure of disc. Once L takes higher values, the velocities rise but the density diminishes. Moreover, we see the L parameter has remarkable effect on the radial and meridional velocities near the surface. In Figure 2, we show variations of the poloidal components of velocity, density and pressure for some values of γ. As γ becomes smaller, both radial and meridional velocities increase. This is exactly contrary to the behaviour of the density and pressure with respect to γ. These plots also clearly show, the γ parameter has a considerable effect on the density and pressure near the equatorial region, vice versa; the presence of this parameter on the poloidal velocities is more noticeable at the surface of the disc. Figure 3 displays radial and meridional components of velocity for different values of ξ. When ξ parameter becomes larger, the poloidal components of velocity decrease in a given polar angle θ and magnitude of this reduction is more significant near the surface of the disc.

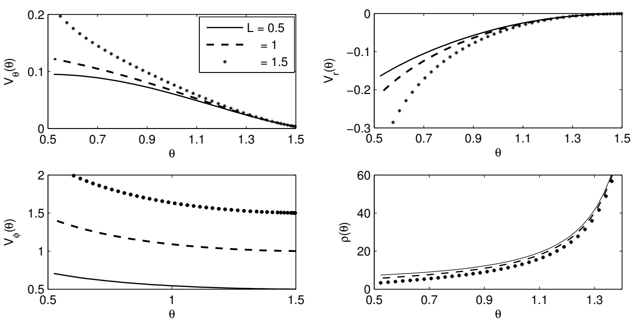

Figure 1. Time dependent self-similar solutions for a thick disc corresponding to $\xi = 0.5$, $\gamma = 5/3$ and several values of L as shown in the key.

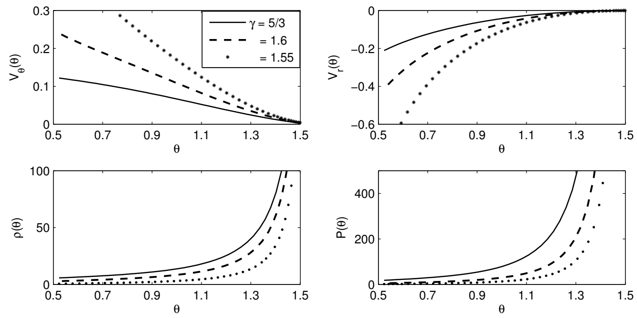

Figure 2. Time dependent self-similar solutions for a thick disc corresponding to $L = \xi = 0.5$ and several values of γ as shown in the key.

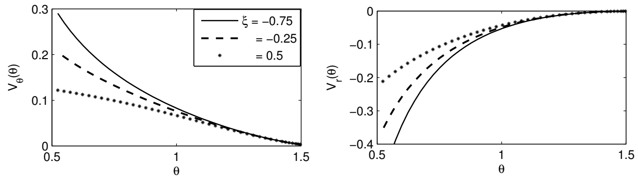

Figure 3. Time dependent self-similar solutions for a thick disc corresponding to $L = 1$, $\gamma = 5/3$ and several values of ξ as shown in the key.

In order to have a clearer picture of the velocity distribution, it is useful to express the results in a Cartesian frame ($y = r\sin\theta$, $z = r\cos\theta$) through the usual relations. We obtain

$$V_y = V_r\sin\theta + V_\theta\cos\theta = -V_\theta\cos\theta + V_\theta\cos\theta = 0,$$

$$V_z = V_r\cos\theta - V_\theta\sin\theta = -V_\theta\cot\theta\cos\theta - V_\theta\sin\theta.$$

Velocity field has been shown in Figure 4. As an important result, while magnetofluid rotates super-Keplerian in the surface of the disc and tends to become Keplerian toward the equator (see equation 50), the net of radial and meridional velocities show magnetofluid has no accretion on the central compact object and only collapses on the equator. Indeed, such velocity distribution causes Lorentz and inertia forces have no any role in the force balance and only the forces of the pressure gradient, gravitation and centrifugal are remained in the equation of force balance (52). On the other hand, as mentioned before, we also expect contribution of pressure

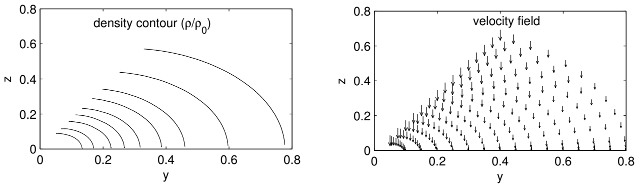

Figure 4. The density contours and velocity field with $z = \frac{r}{r_d} cos\theta,\ y = \frac{r}{r_d} sin\theta$. The constant values are γ = 5/3 and L = 1.

gradient to be insignificant in the radial force balance due to the Keplerian rotation. This situation can be provided only when the pressure gradient decreases rapidly with distance than gravity. We can show

$$\frac{\partial P}{\partial r}\,\alpha\, r^{-\frac{2\gamma-1}{\gamma-1}}.$$

For γ = 5/3, the pressure gradient is proportional to $r^{-3.5}$ and for γ < 5/3 it is decreased more slowly than $r^{-3.5}$. Therefore, in radial direction, the pressure gradient decreases faster than the gravity and we can claim that radial pressure gradient is negligible. However, the pressure gradiant is essential to maintain the disc structure in the latitudinal direction. This analysis is similar to what has been used by Kaburaki (1986) for a magneto-Keplerian disc.

6. Conclusion

The main aim of this work is to obtain a time dependent, self-similar solution for a thick disc surrounding a non-rotating compact object. To this end, we have derived the Newtonian limit of the relativistic time dependent equations for a plasma disc having only a poloidal magnetic field. We have restricted ourselves to fluid flows in which self-gravity and shear viscosity are negligible. We have included all the three fluid velocity components and have considered the plasma to be resistive. A finite resistivity is essential for a disc in the absence of shear viscosity to liberate gravitational energy. The magnetic and electric stresses do not play a role in angular momentum transfer due to the absence of the toroidal components of fields. The angular momentum transfer, therefore, occurs through the finite electrical resistivity of the plasma. Our calculations contain a dimensionless parameter η_0 which specifies the size of the resistivity and can be considered corresponding to α parameter in standard model SS73.

In a self-similar solution, each physical quantity retains a similar spatial form as the flow evolves, but the characteristic length-scale of the flow increases proportionally to $t^{2/3}$. Our results are different from the steady solutions achieved by TPD90 in various respects:

- We find a quasi-keplerian behavior as $\sqrt{\frac{GM}{rsin\theta}}$ for azimuthal velocity, rsinθ f while the azimuthal velocity has sub-Keplerian behavior in stationary disc.
- Due to the non-existence of the toroidal component of the magnetic field, the poloidal currents are vanished ($J^r = J^\theta = 0$). However, in steady state, magnetofluid had a non-zero current in azimuthal direction which is produced due to the motion of magnetofluid and modifies the magnetic field structure inside the disc. The azimuthal current is related to distribution of plasma velocity by ohm law. In current solution, the distribution of velocity ($V_r + cot\theta V_\theta = 0$) gives rise to elimination of azimuthal current($J^\phi = 0$). Therefore, in this sense, magnetofluid is current free. On the other hand, the azimuthal current is also related to the magnetic field by Maxwell equation. The consistency of equations set a lower bound on the conductivity of the fluid (equation 49).
- The configuration of poloidal magnetic field is obtained similar to the stationary disc. However, spacial behavior of the magnetic field, in current time dependent solution, is determined by the $\in_\rho$ parameter which shows density changes with time.
- Because the radial velocity is obtained proportional to the meridional velocity, inertia force $\left(\rho\frac{\partial V}{\partial t} + \rho(V.\nabla)V\right)$ and the Lorentz force J × B are vanished and does not play any role in the force balance. So, in the equation of force-balance, only the terms of the pressure gradi- ent, gravitation and centrifugal are remained (equation 52). In other words, the pressure gradient equals to difference between centrifugal and gravitation terms (see equation 52). In radial direction, Keplerian behavior needs the centrifugal force to be completely balanced by the gravitational force, and thus the pressure gradient is negligible. However, in vertical direction, the pressure gradient is essential to maintain the disc structure. These results reveal that magnetofluid has no accretion on compact object and is a rotating disc. It is exactly different from steady accretion flow.

The previous studies on one-dimensional time dependent systems Mineshige & Umemura[25]; Khesali &

Faghei[18,19] exhibited that the radial behaviour of physical quantities was different from the results achieved by those who considered stationary states. Our results, in addition, show that time dependence can vary the disc structure.

7. References

1. Abbassi S, Ghanbari J, Nahhar S, MNRAS. 2008; 388(2):663.
2. Abbassi S, Ghanbari J, Ghasemnezhad M. MNRAS. 2010; 409:1113.
3. Akizuki C, Fukue J. PASJ. 2006; 58:469.
4. Arai K, Hashimoto M. A&A. 1992; 254:191.
5. Banerjee D, Bhatt JR, Das AC, Prasanna AR. ApJ. 1995; 449:789.
6. Biermann M, Duschl WJ. AGM. 2000; 17:A15.
7. Bostrom R. Ap&SS. 1973; 22:353.
8. Davidson PA. An Introduction to Magnetohydrodynamics. Cambrige University Press; 2001.
9. Faghei K. MNRAS. 2012; 420:118.
10. Frank J, King A, Raine D. Accretion Power in Astrophysics. Cambridge: Cambridge Univ Press; 1992.
11. Fukue J, Sakamoto C. PASJ. 1992; 44:553.
12. Ghanbari J, Abbassi S, MNRAS. 2004; 350:1437.
13. Ghanbari J, Salehi F, Abbassi S. MNRAS. 2007; 381:159.
14. Hashimoto M, Eriguchi Y, Arai K, Muller E. A&A. 1993; 268:131.
15. Hoshi R. PTP. 1979; 61:1307.
16. Igumenshchev IV, Abramowicz MA. MNRAS. 1999; 303:309.
17. Igumenshchev IV, Abramowicz MA. ApJS. 2000; 130:463.
18. Khesali A, Faghei K. MNRAS. 2008; 389:1218.
19. Khesali A, Faghei K. MNRAS. 2009; 398:1361.
20. Kaburaki O. MNRAS. 1987; 229:165.
21. Kaburaki O. ApJ. 2000; 531:210.
22. Lightman AP. APJ. 1974; 194:419.
23. Meyer F, Meyer-Hofmeister E. A&A. 1981; 104:L10.
24. Meszaros P. A&A. 1974; 35:171.
25. Mineshige S, Umemura M. ApJ. 1997; 480:167.
26. Mineshige S, Nakayama K, Umemura M. PASJ. 1997; 49:439.
27. Mosallanezhad A, Abbassi S, Beiranvand N. MNRAS. 2014; 437:3112.
28. Narayan R, Yi I. ApJ. 1994; 428:L13.
29. Narayan R, Yi I. ApJ. 1995; 444:231.
30. Ohsuga K, Mineshige S. ApJ. 2011; 736:2.
31. Ogilvie GI. MNRAS. 1999; 306:9.
32. Prasanna AR, Tripathy SC, Das AC. JApA. 1989; 10:21 (PTD89).
33. Pringle JE. ARA&A. 1981; 19:137.
34. Proga D, Begelman MC. ApJ. 2003; 592:767.
35. Rice WKM, Mayo JH, Armitage PJ. MNRAS. 2010; 402:1740.
36. Ross RR, Fabian AC, Mineshige S. MNRAS. 1992; 258:189.
37. Samadi M, Abbassi S, Khajavi M. MNRAS. 2014; 437:3124.
38. Shadmehri M. A&A. 2004; 424:379.
39. Shadmehri M. APJ. 2004; 612:1000.
40. Ghanbari J, Shaghaghian M. PASJ. 2009; 61:1261.
41. Shaghaghian M. MNRAS. 2011; 415:534.
42. Shakura NI, Sunyaev RA. A&A. 1973; 24:337.
43. Tripathy SC, Prasanna AR, Das AC. MNRAS. 1990; 246:384 (TPD90).
44. Wandel A, Petrosian V. ApJ. 1988; 329:L11.

Appendix A.

Basic equations in Relativistic Framework

Basic equations governing the dynamical evolution of an axisymmetric magnetofluid disc around a static compact object in the general relativistic framework are described by expanding the equations (1)-(2) as the continuity equation

$$\left(\rho+\frac{P}{c^2}\right)\left(1-\frac{2m}{r}\right)\left[\frac{\partial V^{(r)}}{\partial r}+\frac{1}{r}\frac{\partial V^{(\theta)}}{\partial \theta}+\frac{1}{r}\left(2V^{(r)}+V^{(\theta)}cot\theta\right)\right]+\left(1-\frac{2m}{r}\right)V^{(r)}\frac{\partial}{\partial r}\left(\rho-\frac{P}{c^2}\right)$$
$$+\left(1-\frac{2m}{r}\right)\frac{V^{(\theta)}}{r}\frac{\partial}{\partial \theta}\left(\rho-\frac{P}{c^2}\right)+\frac{\partial}{\partial t}\left(\rho-\frac{P}{c^2}\right)+\frac{1}{c^2}\left(1-\frac{2m}{r}\right)\left(1-\frac{V^2}{c^2}\right)\left(F_k^0-2F_{ik}u^iu^0\right)J^k/c=0, \tag{A.1}$$

the radial

$$\left(\rho+\frac{P}{c^2}\right)\left(1-\frac{2m}{r}\right)\left(1-\frac{V^2}{c^2}\right)^{-1}\left[\left(1-\frac{2m}{r}\right)^{-1}\frac{\partial V^{(r)}}{\partial t}+V^{(r)}\frac{\partial V^{(r)}}{\partial r}+\frac{V^{(\theta)}}{r}\frac{\partial V^{(r)}}{\partial \theta}+\frac{MG}{r^2}\left(1-\frac{2m}{r}\right)^{-1}\right.$$
$$\left.\times\left(1-\frac{{V^{(r)}}^2}{c^2}\right)-\frac{1}{r}\left({V^{(\theta)}}^2+{V^{(\varphi)}}^2\right)\right]+\frac{V^{(r)}}{c^2}\frac{\partial P}{\partial t}+\left(1-\frac{2m}{r}\right)\frac{\partial P}{\partial r}+\left(F_k^0\frac{V^r}{c}-F_c^k\right)\frac{J^k}{c}=0, \tag{A.2}$$

the meridional

$$\left(\rho+\frac{P}{c^2}\right)\left(1-\frac{V^2}{c^2}\right)^{-1}\left[\frac{\partial V^{(\theta)}}{\partial t}+\left(1-\frac{2m}{r}\right)^{1/2}V^r\frac{\partial V^{(\theta)}}{\partial r}+\frac{V^\theta}{r}\left(1-\frac{2m}{r}\right)^{1/2}\frac{\partial V^{(\theta)}}{\partial \theta}+\frac{1}{r}\left(1-\frac{3m}{r}\right)\right.$$
$$\left.\times\left(1-\frac{2m}{r}\right)^{-1/2}-V^{(\theta)}V^{(r)}-\frac{1}{r}cot\theta V^{(\varphi)^2}\right]+\frac{1}{r}\frac{\partial P}{\partial \theta}+\left(F_k^0\frac{V^\theta}{c}-F_k^0\right)\frac{J^k}{c}=0, \quad \text{(A.3)}$$

and the azimuthal component of momentum equation

$$\left(\rho+\frac{P}{c^2}\right)\left(1-\frac{V^2}{c^2}\right)^{-1}\left[\frac{\partial V^{(\varphi)}}{\partial t}+\left(1-\frac{2m}{r}\right)V^{(r)}\frac{\partial V^{(\varphi)}}{\partial r}+\frac{V^\theta}{r}\frac{\partial V^{(\varphi)}}{\partial \theta}\left(1-\frac{2m}{r}\right)+\frac{1}{r}\left(1-\frac{3m}{r}\right)V^{(\varphi)}V^{(r)}\right.$$
$$\left.+\frac{1}{r}\left(1-\frac{2m}{r}\right)cot V^{(\varphi)}V^{(\varphi)}\right]+\frac{V^{(\varphi)}}{c^2}\frac{\partial P}{\partial t}+\left(F_k^0\frac{V^\varphi}{c}-F_k^\varphi\right)\frac{J^k}{c}=0, \quad \text{(A.4)}$$

and the Maxwell equations

$$J^{(r)}=-\frac{c}{4\pi rsin\theta}\left[\frac{\partial}{\partial\theta}\left(sin\theta B_{(\varphi)}\right)-rsin\,\theta\left(1-\frac{2m}{r}\right)^{-1/2}\frac{\partial E_{(r)}}{\partial t}\right] \quad \text{(A.5)}$$

$$J^{(\theta)}=\frac{c}{4\pi r}\left\{\frac{\partial}{\partial r}\left[\left(1-\frac{2m}{r}\right)^{1/2}rB_{(\varphi)}\right]+\frac{1}{r}\left(1-\frac{2m}{r}\right)^{-1/2}\frac{\partial E_{(\theta)}}{\partial t}\right\} \quad \text{(A.6)}$$

$$J^{(\varphi)}=\frac{c}{4\pi r}\left\{\frac{\partial}{\partial r}\left[\left(1-\frac{2m}{r}\right)^{1/2}rB_{(\theta)}\right]-\frac{\partial B_{(r)}}{\partial\theta}+\frac{1}{rsin\theta}\left(1-\frac{2m}{r}\right)^{-1}\frac{\partial E_{(\varphi)}}{\partial t}\right\} \quad \text{(A.7)}$$

$$J^{(t)}=\frac{c}{4\pi r^2 sin\theta}\left(1-\frac{2m}{r}\right)^{1/2}\left\{\frac{\partial}{\partial r}\left(r^2 sin\theta E_{(r)}\right)+\frac{\partial}{\partial\theta}\left[E_{(\theta)}rsin\theta\left(1-\frac{2m}{r}^{-1/2}\right)\right]\right\} \quad \text{(A.8)}$$

$$\frac{\partial}{\partial\theta}\left[rsin\left(1-\frac{2m}{r}\right)^{-1/2}B_{(\theta)}\right]+\frac{\partial}{\partial r}\left(r^2 sin\theta B_{(r)}\right)=0 \quad \text{(A.9)}$$

$$\frac{\partial}{\partial t}\left[\left(1-\frac{2m}{r}\right)^{-1/2}rB_{(\varphi)}\right]+\frac{\partial}{\partial r}\left[\left(1-\frac{2m}{r}\right)^{1/2}rE_{(\theta)}\right]-\frac{\partial E_{(r)}}{\partial\theta}=0 \quad \text{(A.10)}$$

$$-\frac{\partial}{\partial\theta}\left[rsin\theta\left(1-\frac{2m}{r}\right)^{-1/2}B_{(\theta)}\right]+\frac{\partial}{\partial r}\left[rsin\theta\left(1-\frac{2m}{r}\right)^{1/2}E_{(\theta)}\right]=0 \quad \text{(A.11)}$$

$$\frac{\partial}{\partial t}\left(r^2 sin\theta B_{(r)}\right)+\frac{\partial}{\partial\theta}\left[rsin\theta\left(1-\frac{2m}{r}\right)^{1/2}E_{(\varphi)}\right]=0. \quad \text{(A.12)}$$

The parentheses around an index indicate the local Lorentz component.