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Ranking Method Based on Cross-efficiency and Aggregate Units in Data Envelopment Analysis

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Abstract

In this study a new method is developed for ranking all type of efficient decision making units (extreme and non-extreme ones) which is based on cross-efficiency aggregate units. Also, this study is able to encountering alternatives in order to find the best solutions among alternatives.

Keywords: Aggregate Unit, Cross-Efficiency, Data Envelopment Analysis, Ranking

1. Introduction

Before Data Envelopment Analysis (DEA) has been developed, Charnes et al.[4] it had been initialized by Farrel[7] as a non-parametric method to evaluate the relative efficiency of different organizations. The DEA as the conventional framework has been used as a mathematical programming tool to observational data or equivalently, Decision Making Units (DMUs), to provide an important definition called the Production Possibility Set (PPS). Also, it is substantial to introduce the production function used to assess DMUs. In the DEA context, the relative efficiency of DMUs is attained via comparing them with the efficiency frontier. The DEA has been known with the original CCR model, and then different theoretical extensions have been developed such as: the BCC model, Banker et al.[3] is a variation with variable returns-to-scale. Ranking DMUs is as a field that many papers[1] have been adapted to it. DEA just by consideration to relative efficiency scores of DMUs achieved by solving conventional models dichotomizes DMUs into two distinct groups: efficient DMUs and inefficient ones[5]. Unfortunately, the DEA despite of its popularity in different contexts cannot provide adequate information to discriminate among efficient DMUs which have the equal efficiency value, namely one. In the DEA models in order to evaluate the relative efficiency, each DMU is assigned to the best weights. Since these weights are different from one DMU to another one, the obtained efficiency scores are non-comparable, and the efficient DMUs do not necessarily have the same performances in actual practices. Therefore, recently many papers[8,12] have been assigned in the field of ranking. From the cross-efficiency ranking method[11] as the initial working up to now, many various extensions have been brought into the DEA context, such as the super-efficiency (AP)[2] variation, the Sinuany-Stern's variation[12] that applies multivariate statistical tools in order to obtain a complete ranking to DMUs. The Common Set of Weights method (CSW)[6,10] are as another significant ranking approaches by which a Decision Maker (DM) encounters a group of assessments such as branches of a bank. The rest of the paper is organized as follows. In Section 2, the preliminary of DEA is presented. In section 3, the proposed ranking system is introduced. In section 4, an approach is developed to find the best optimal weights among alternatives. A numerical example is included in section 5, and finally conclusions are presented in section 6.

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2. The Preliminary of DEA

The DEA is a mathematical approach by which the variable weights are derived directly from the data. Let there are n DMUs and the assessed DMU to be DMU_p whose the given value of indices are denoted as $(x_{1p}, x_{2p}, \ldots, x_{mp})$ and $(y_{1p}, y_{2p}, \ldots, y_{sp})$, respectively. It should be noted that we need to solve the following fractional programming problem 1 or the equivalent linear form 2 once to measure the best efficiency value of DMU_p. Now, let (u_p^*, u_p^*) be the vector of optimal weight to the DMU_p in the sense of maximizing the ratio scale, $\theta_p^* = \frac{\sum_{r=1}^{s} u_{rp}^* y_{rp}}{\sum_{i=1}^{m} v_{ip}^* x_{ip}}$ obtained via the following model 1:

$$
\begin{aligned}
&FP_p: \\
&\theta_p^* = \max \quad \frac{\sum_{r=1}^{s} u_{rp} y_{rp}}{\sum_{i=1}^{m} u_{ip} x_{ip}} \\
&s.t. \quad \frac{\sum_{r=1}^{s} u_{rp} y_{rj}}{\sum_{i=1}^{m} v_{ip} x_{ij}} \le \quad j = 1, 2, \ldots, n \\
&\quad u_{rp} \ge \varepsilon > 0, \quad r = 1, 2, \ldots, s, \\
&\quad v_{ip} \ge \varepsilon > 0, \quad i = 1, 2, \ldots, m.
\end{aligned}
\tag{1}
$$

Or the equivalent linear form 2 as follows:

$$
\begin{aligned}
&LP_p: \\
&\theta_p^* = \max \quad \sum_{r=1}^{s} u_{rp} y_{rp} \\
&s.t. \quad \sum_{i=1}^{m} v_{ip} x_{ip} = 1, \\
&-\sum_{i=1}^{m} v_{ip} x_{ij} + \sum_{r=1}^{s} u_{rp} y_{rj} \le 0, \quad j = 1, 2, \ldots, n \\
&\quad u_{rp} \ge \varepsilon > 0, \quad r = 1, 2, \ldots, s, \\
&\quad v_{ip} \ge \varepsilon > 0, \quad i = 1, 2, \ldots, m.
\end{aligned}
\tag{2}
$$

Where, (u_p, v_p) is a weight vector. The DMU_p is efficient if $\theta_p^* = 1$ in the model 2 and otherwise, is inefficient. Since this does not mean that all efficient DMUs have an equivalent performance, so a true judgment about prioritizing among efficient DMUs needs some additional information derived by adapting ranking methods. In this study a cross-evaluation method is proposed to make a complete ranking of all types of efficient DMUs.

3. Main Formulation

In this section a ranking index to evaluate performance of all efficient DMUs is introduced by cross-evaluating values related to the efficiency of some virtual units called aggre gate units defined later. At first, some signs should be defined. Let $E = \{DMU_j / \theta_j^* = 1.0;\ j = 1, \ldots, n\}$ be as the efficient set and $J = \{j / DMU_j \in E\}$ be an index set related to E. Suppose $DMU_{\bar{a}}$ be the sign of the aggregate unit. The $DMU_{\bar{a}}$ is defined by $(x_{\bar{a}}, y_{\bar{a}})$ whose they are the input and output vector, respectively. In fact they are as the aggregate input and output vector defined over all input and output indices of all efficient DMUs as follows:

$$
\begin{aligned}
x_{i\bar{a}} &= \sum_{j \in J} x_{ij}, \quad i = 1, 2, \ldots, m. \\
y_{r\bar{a}} &= \sum_{j \in J} y_{ij}, \quad r = 1, 2, \ldots, s.
\end{aligned}
\tag{3}
$$

Obviously, an efficient DMU would be more preferred or would be a better-performance efficient DMU, if it produces more outputs by consuming less inputs in comparison with the other efficient DMUs. Regarding the equation 3, it is concluded that input vector of a better performance efficient DMU should be contribute to the $x_{\bar{a}}$ weakly, and conversely, its output vector should contribute to the $y_{\bar{a}}$ strongly. Then if such better-performance efficient DMU, namely DMU_p, is deleted from the set E, the $x_{\bar{a}}$ should lose less volume of its amounts, but $y_{\bar{a}}$ should lose more volume of its amounts. To show this statement, suppose all efficient DMUs as an aggregate unit $DMU_{\bar{a}}$ try to reach the maximum efficiency score, namely one, by adopting some appropriate weights via the following model:

$$
\begin{aligned}
&FP_{DMU_{\bar{a}}}: \\
&\theta_{\bar{a}}^* = \max \quad \frac{\sum_{r=1}^{s} u_{r\bar{a}} y_{r\bar{a}}}{\sum_{i=1}^{m} v_{i\bar{a}} x_{i\bar{a}}} \\
&s.t. \quad \frac{\sum_{r=1}^{s} u_{i\bar{a}} y_{r\bar{a}}}{\sum_{i=1}^{m} v_{i\bar{a}} x_{i\bar{a}}} \le \quad 1, \\
&u_{r\bar{a}} \ge \varepsilon > 0, \quad r = 1, 2, \ldots, s, \\
&v_{i\bar{a}} \ge \varepsilon > 0, \quad i = 1, 2, \ldots, m.
\end{aligned}
\tag{4}
$$

Or via the equivalent linear form as below:

$$
\begin{aligned}
&LP_{DMU_{\bar{a}}}:\\
&\theta^*_{\bar{a}} = \max \quad \sum_{r=1}^{s} u_{r\bar{a}} y_{r\bar{a}}\\
&s.t. \quad \sum_{i=1}^{m} v_{i\bar{a}} x_{i\bar{a}} = 1,\\
&\qquad -\sum_{i=1}^{m} v_{i\bar{a}} x_{i\bar{a}} + \sum_{r=1}^{s} u_{r\bar{a}} y_{r\bar{a}} \le 0,\\
&u_{r\bar{a}} \ge \varepsilon > 0, \quad r = 1,2,\dots,s,\\
&v_{i\bar{a}} \ge \varepsilon > 0, \quad i = 1,2,\dots,m.
\end{aligned} \tag{5}
$$

Where $u_{r\bar{a}}, r = 1,\dots,S$ and $v_{i\bar{a}}, i = 1,\dots,m$ are as output weights and input weights, respectively.

After solving the model, let $(u^*_{1\bar{a}}, u^*_{2\bar{a}},\dots,u^*_{s\bar{a}}, v^*_{1\bar{a}}, v^*_{2\bar{a}},\dots,v^*_{m\bar{a}})$ be as the most proper optimal weights among all optimal alternative weight (solutions). Then, in order to evaluate the performance of the DMU_p, it is deleted from the set E, then based on the remaining efficient DMUs, a new aggregate DMU, namely $DMU^p_{\bar{a}}$, is defined as equation 6:

$$
\begin{aligned}
x^p_{i\bar{a}} &= \sum_{j\in J_p} x_{ij}, \qquad i = 1,2,\dots,m,\\
y^p_{r\bar{a}} &= \sum_{j\in J_p} y_{rj}. \qquad r = 1,\dots,s.
\end{aligned} \tag{6}
$$

Where $J_p = \{j / DMU_j \in E_p\}$ and $E_p = E \setminus \{DMU_p\}$

Now the following model (like the model 5) is adopted to maximize the efficiency score of the $DMU^p_{\bar{a}}$ as:

$$
\begin{aligned}
&LP_{DMU^p_{\bar{a}}}:\\
&\theta^{p*}_{\bar{a}} = \max \quad \sum_{r=1}^{s} u^p_{r\bar{a}} y^p_{r\bar{a}}\\
&s.t. \quad \sum_{i=1}^{m} v^p_{i\bar{a}} x^p_{i\bar{a}} = 1,\\
&\qquad -\sum_{i=1}^{m} v^p_{i\bar{a}} x^p_{i\bar{a}} + \sum_{r=1}^{s} u^p_{r\bar{a}} y^p_{r\bar{a}} \le 0,\\
&u^p_{r\bar{a}} \ge \varepsilon > 0, \quad r = 1,2,\dots,s,\\
&v^p_{i\bar{a}} \ge \varepsilon > 0, \quad i = 1,2,\dots,m.
\end{aligned} \tag{7}
$$

It must be noted that based on the DMU_p's performance, the volume of $x^p_{\bar{a}}$ and $y^p_{\bar{a}}$ would be different. As noted earlier, if DMU_p is to be better- performance efficient DMU, i.e. it produces more outputs by using less inputs, after its deletion, the $y_{\bar{\alpha}}$ loses more volume of its amounts and conversely, $x_{\bar{\alpha}}$ loses a less volume of its amounts. Therefore, the $DMU^p_{\bar{a}}$ in order to reach its maximum efficiency score, namely one, must apply less input weights and more output weights. So, let $(u^{p*}_{1\bar{a}}, u^{p*}_{2\bar{a}},\dots,u^{p*}_{s\bar{a}}, v^{p*}_{1\bar{a}}, v^{p*}_{2\bar{a}},\dots,v^{p*}_{m\bar{a}})$ be as a proper weight vector obtained from the model 7. In continue, a new approach is developed to select the most proper optimal solutions among all optimal alternatives in the models 5 or 7 for assessing DMU_p. To this end, we have:

$$
CEI_{kl} = \frac{\dfrac{\sum_{r=1}^{s} u^k_{r\bar{a}}}{\sum_{i=1}^{s} u^l_{r\bar{a}}}}{\dfrac{\sum_{i=1}^{m} v^k_{i\bar{a}}}{\sum_{i=1}^{m} v^l_{i\bar{a}}}}, \qquad k,l \in \bar{J}. \tag{8}
$$

Or

$$
\frac{\left(\sum_{r=1}^{s} u^k_{r\bar{a}}\right)\left(\sum_{i=1}^{m} v^l_{r\bar{a}}\right)}{\left(\sum_{r=1}^{s} u^l_{r\bar{a}}\right)\left(\sum_{i=1}^{m} v^k_{i\bar{a}}\right)}, \quad k,l \in \bar{J}.
$$

where $\bar{J} = JU\{\bar{a}\}$.

Equation 8 shows the performance of DMU_k in comparison with DMU_l, respectively.

Definition: Now the proposed ranking index based on cross-efficiency and aggregate units is defined as follows:

$$
CEI_k = \frac{\sum_{l\in\bar{J}} CEI_{kl}}{\sum_{l\in\bar{J}} CEI_{\bar{a}l}}, \ k \in J. \tag{9}
$$

4. Finding the Best Optimal Solutions

It must be noted that the models 5 and/or 7 may encounter alternative optimal weighs. Since different optimal weights result in different ranking results, developing an approach to select the most proper optimal solutions among all alternative optimal weights would be significant. Therefore, in this section an approach comprising (m+s) linear models is defined to clarify the best optimal weights among alternatives. The idea of this approach is derived from two subjects: 1. Based on the idea of Obata et al. in which adopting smaller output weights and conversely larger input weights is preferable. 2. Adopting minimum output weights and maximum input weights[9] at least changes would be imposed on members of E_p after deleting of $DMU_{p.}$

Based on the two above subjects, the proposal is as follows:

$$u_{1\bar{a}}^{p^*} = \min \quad u_{1\bar{a}}^{p}, \qquad (10-O-1)$$
$$s.t. \quad \sum_{i=1}^{m} u_{r\bar{a}}^{p} y_{r\bar{a}}^{p} = 1$$
$$u_{r\bar{a}}^{p} \geq \varepsilon > 0, \quad r = 1,\ldots,s,$$
$$\vdots$$
$$u_{s\bar{a}}^{p^*} = \min \quad u_{s\bar{a}}^{p}, \qquad (10-O-s)$$
$$s.t. \quad u_{1\bar{a}}^{p} = u_{1\bar{a}}^{p^*},$$
$$u_{2\bar{a}}^{p} = u_{2\bar{a},}^{p^*},$$
$$\vdots$$
$$u_{(s-1)\bar{a}}^{p} = u_{(s-1)\bar{a},}^{p^*}$$
$$\sum_{r=1}^{s} u_{r\bar{a}}^{p}\, y_{r\bar{a}}^{p} = 1,$$
$$u_{r\bar{a}}^{p} \geq \varepsilon > 0, \qquad i = 1,\ldots,s.$$

$$v_{1\bar{a}}^{p^*} = \max \quad v_{1\bar{a}}^{p}, \qquad (10-I-1)$$
$$s.t. \quad \sum_{i=1}^{m} v_{i\bar{a}}^{p} x_{i\bar{a}}^{p} = 1$$
$$v_{i\bar{a}}^{p} \geq \varepsilon > 0, \quad i = 1,\ldots,m,$$
$$\vdots$$
$$v_{m\bar{a}}^{p^*} = \max \quad v_{m\bar{a}}^{p}, \qquad (10-I-m)$$
$$s.t. \quad v_{1\bar{a}}^{p} = v_{1\bar{a}}^{p^*},$$
$$v_{2\bar{a}}^{p} = v_{2\bar{a},}^{p^*},$$
$$\vdots$$
$$v_{(m-1)\bar{a}}^{p} = v_{(m-1)\bar{a},}^{p^*}$$
$$-\sum_{i=1}^{m} v_{i\bar{a}}^{p}\, x_{i\bar{a}}^{p} = 1,$$
$$v_{i\bar{a}}^{p} \geq \varepsilon > 0, \qquad i = 1,\ldots,m.$$

5. Numerical Example

As mentioned before, this technique has some benefits and in order to show them, one numerical example with real data extracted from[1] has been shown as below.

Example: There are six DMUs which are compared over four variables: let Staff Hours Per Day (StHr) and Supplies Per Day (Supp) be as the inputs, and total Medicare Plus Medicaid Reimbursed Patient Days (MCPD) and total Private Patient Days (PPPD) be as the outputs, which are shown in Table 1 and our proposal's results and some other ranking methods generated the results presented in Table 2.

Table 1. DMU's data (extracted from[1] (p 260))

DMU	Input 1	Input 2	Output 1	Output 2
a	150.000	0.200	14000.000	3500.000
b	400.000	0.700	14000.000	21000.000
c	320.000	1.200	42000.000	10500.000
d	520.000	2.000	28000.000	42000.000
e	350.000	1.200	19000.000	25000.000
f	320.000	0.700	14000.000	15000.000

Table 2. DMU's score to the proposed system and some other ranking methods

CEI		CCR		BCC		CEA		CEB	
a	1.142	a	1.000	a	1.000	a	1.000	a	1.000
c	1.07	b	1.000	b	1.000	b	0.916	d	1.000
b	0.997	c	1.000	c	1.000	d	0.916	b	0.955
d	0.769	d	1.000	d	1.000	c	0.842	c	0.886

6. Conclusion

In this paper a cross-evaluation ranking index shown by $CEI_p(p \in J)$ was defined. The rest of the paper was that in section 2, an introduction of the DEA was displayed, briefly. In section 3, the propose model was discussed. In section 4, in order to find the best optimal weights among alternatives, an approach including (m+s) linear models was suggested. In section 5, a numerical example was displayed, and also the proposed model was compared with some other traditional ranking models.

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