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Wavelet Solution for Class of Nonlinear Integro-differential Equations

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Abstract

The aim of this work is to study the Legendre wavelets for the solution of a class of nonlinear Volterra integro-differential equation. The properties of Legendre wavelets together with the Gaussian integration method are used to reduce the problem to the solution of nonlinear algebraic equations. Also a reliable approach for convergence of the Legendre wavelet method when applied to nonlinear Volterra equations is discussed. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique and the results obtained by Legendre wavelet method is very nearest to the exact solution. The results demonstrate reliability and efficiency of the proposed method.

Keywords: Legendre Wavelets, Integro-differential Equations, Gaussian Integration, Legendre Wavelet Method, Convergence Analysis.

1. Introduction

Integro-Differential Equation (IDE) is an equation that the unknown function appears under the sign of integration and it also contains the derivatives of the unknown function. Mathematical modeling of real-life problems usually results in functional equations, e.g. partial differential equations, integral and IDE, stochastic equations and others. Many mathematical formulations of physical phenomena contain IDE; these equations arise in fluid dynamics, biological models and chemical kinetics. In the past several decades, many effective methods for obtaining approximation/numerical solutions of linear/nonlinear differential equations have been presented, such as Adomian decomposition method [1], Variational iteration method [13, 15], Homotopy perturbation method [1, 9, 10, 11, 12], He's Homotopy perturbation method [2, 3, 7, 12], Homotopy analysis method [14], wavelet method [4, 5, 6, 8, 11, 16] etc.,

Ghasemi et al. [12] presented He's homotopy perturbation method for solving nonlinear integro differential equations. Zhao and Corless [25] adopted finite difference method for integro-differential equations. Yusufoglu et al. [24] had solved initial value problem for Fredholm type linear integro-differential equation system. Seyed Alizadeh et al [18] discussed an Approximation of the Analytical Solution of the Linear and Nonlinear Integro-Differential Equations by Homotopy Perturbation Method. Wazwaz [23] gave a reliable algorithm for solving boundary value problems for higher-order integro-differential equations. Lepik [16] had solved the nonlinear integro differential equations using Haar wavelet method. Ghasemi, et al. [9] discussed the comparison between wavelet Galerkin method and homotopy perturbation method for the nonlinear integro-differential equations. Ghasemi et al. [8, 11] established numerical solution of linear integro-differential equations by using sine-cosine wavelet method and they have also compared with homotopy perturbation method.

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In recent years, wavelets have found their way in to many different fields of science and engineering. Many researchers started using various wavelets for analyzing problems of greater computational complexity and proved wavelets to be powerful tools to explore new direction in solving differential equations. Recently, Venkatesh et al. [19, 20, 21, 22] applied Legendre wavelets for the solution of initial value problems of Bratu-type and higher order Volterra IDE, Cauchy problems and they have also discussed theoretical analysis of Legendre wavelets method for the solution of Fredholm integral equations.

In the present article, we apply Legendre wavelet method to find the approximate solution of

$$u'(x)=f(x)+\int_0^x g\left(t,u(t),u'(t)\right)dt \tag{1}$$

with the initial condition $u(0)=s$ (a constant) where $f(x)$ is the source term. The Legendre wavelet method (LWM) consists of conversion of integro-differential equations into integral equations and expanding the solution by Legendre wavelets with unknown coefficients. The properties of Legendre wavelets together with the Gaussian integration formula are then utilized to evaluate the unknown coefficients and find an approximate solution to Eq. (1).

The organization of the paper is as follows: In section 2, we describe the basic formulation of wavelets and Legendre wavelets required for our subsequent development. Section 3 is devoted to the solution of (1) by using integral operator and Legendre wavelets. Convergence analysis and the error estimation for the proposed method have been discussed in section 4. In section 5, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples. Concluding remarks are given in the final section.

2. Properties of Legendre Wavelets

2.1 Wavelets and Legendre Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter 'a' and the translation parameter 'b' vary continuously, we have the following family of continuous wavelets as:

$$\psi_{a,b}(t)=|a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right),\ a,b\in R,\ a\neq 0.$$

If we restrict the parameters a and b to discrete values as $a=a_0^{-k}$, $b=n\,b_0\,a_0^{-k}$, $a_0>1$, $b_0>0$ and n, and k positive integer, we have the following family of discrete wavelets: $\psi_{k,n}(t)=|a|^{-\frac{1}{2}}\psi\left(a_0^{\,k}t-nb_0\right)$ where $\psi_{k,n}(t)$ form a basis of $L^2(R)$. In particular, when $a_0=2$ and $b_0=1$ then $\psi_{k,n}(t)$ forms an orthonormal basis.

Legendre wavelets $\psi_{nm}(t)=\psi\left(k,\hat{n},m,t\right)$ have four arguments: $\hat{n}=2n-1$, $n=1,2,3\ldots,2^{k-1}$, k can assume any positive integer, m is the order for Legendre polynomials and t is the normalized time. They are defined on the interval [0,1] as

$$\psi_{nm}(t)=\begin{cases}\sqrt{m+\frac{1}{2}}\,2^{\frac{k}{2}}P_m\left(2^k t-\hat{n}\right), & for\ \frac{\hat{n}-1}{2^k}\le t\le\frac{\hat{n}+1}{2^k},\\ 0 & ,otherwise\end{cases} \tag{2}$$

where $m=0,1,2,\ldots,M-1$, $n=1,2,3,\ldots,2^{k-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is $a=2^{-k}$ and translation parameter is $b=\hat{n}2^{-k}$.

Here $P_m(t)$ is well-known Legendre polynomials of order m which are defined on the interval [−1,1], and can be determined with the aid of the following recurrence formulae:

$$P_o(t)=1\ ,\qquad P_1(t)=t$$

$$P_{m+1}(t)=\left(\frac{2m+1}{m+1}\right)t\,P_m(t)-\left(\frac{m}{m+1}\right)P_{m-1}(t)\ ,\quad m=1,2,3,\ldots.$$

2.2 Function Approximation

A function $f(t)$ defined over [0,1) may be expanded as

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\,\psi_{nm}(t) \tag{3}$$

where $c_{nm}=\left(f(t),\psi_{nm}(t)\right)$, in which (. , .) denotes the inner product. If the infinite series in Eq.(3) is truncated, then Eq.(3) can be written as

$$f(t)\cong\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{nm}\,\psi_{nm}(t)=C^T\psi(t),$$

where C and $\psi(t)$ are $2^{k-1}M$ x 1matrices given by

$$C=\left[c_{10},c_{11},\ldots c_{1M-1},c_{20},\ldots,c_{2M-1},\ldots,c_{2^{k-1}0},\ldots,c_{2^{k-1}M-1}\right]^T, \tag{4}$$

$$\psi(t)=\begin{bmatrix}\psi_{10}(t),\psi_{11}(t),\ldots,\psi_{1M-1}(t),\psi_{20}(t),\ldots,\\ \psi_{2M-1}(t),\ldots,\psi_{2^{k-1}0}(t),\ldots,\psi_{2^{k-1}M-1}(t)\end{bmatrix}^T. \tag{5}$$

3. Legendre Wavelet Scheme for Nonlinear Integro-differential Equations

Consider the integro differential equation given in Eq.(1)

Integrating Eq.(1) w.r.t 'x' both sides, we get

$$u(x)=s+\int_0^x f(x)\,dx+\int_0^x\int_0^x g(t,u(t),u'(t))\,dt\;dx$$

$$u(x)=G(x)+\int_0^x F(t,u(t))\,dx \tag{6}$$

where $G(x)=s+\int_0^x f(x)dx$ and

$F(x,u(x))=\int_0^x g(t,u(t),u'(t))\,dt$

Let $u(x)=C^T\,\psi(x)$. (7)

Therefore we have $C^T\,\psi(x)=G(x)+\int_0^x F\left(t,C^T\,\psi(t)\right)dx.$ (8)

We now collocate Eq.(8) at $2^{k-1}M$ points at x_i as

$$C^T\,\psi(x_i)=G(x)+\int_0^{x_i} F\left(C^T\,\psi(x_i)\right)dx. \tag{9}$$

Suitable collocation points are zeros of Chebyshev polynomials

$$x_i=\cos\left((2i+1)\pi/2^k M\right),\quad i=1,2,\ldots,2^{k-1}M.$$

In order to use the Gaussian integration formula for Eq. (9), we transfer the intervals $[0,x_i]$ into the interval $[-1,1]$ by means of the transformation

$$\tau=\frac{2}{x_i}t-1.$$

Eq. (9) may then be written as

$$C^T\,\psi(x_i)=G(x_i)+\frac{x_i}{2}\int_{-1}^{1}F\left(\frac{x_i}{2}(\tau+1),\,C^T\,\psi\left(\frac{x_i}{2}(\tau+1)\right)\right)d\tau$$

By using the Gaussian integration formula, we get

$$C^T\,\psi(x_i)\approx G(x_i)+\frac{x_i}{2}\sum_{j=1}^{s}w_j\,F\left(\frac{x_i}{2}(\tau+1),C^T\,\psi\left(\frac{x_i}{2}(\tau+1)\right)\right),$$
$$i=1,2,\ldots,2^{k-1}M, \tag{10}$$

where τ is s zeros of Legendre polynomials P_{s+1} and w_j are the corresponding weights. The idea behind the above approximation is the exactness of the Gaussian integration formula for polynomials of degree not exceeding $2s+1$. Eq. (10) gives $2^{k-1}M$ nonlinear equations which can be solved for the elements of C in Eq. (7) using Newton's iterative method.

4. Convergence Analysis

In this section, we discuss the convergence and error analysis of our proposed method.

Theorem 4.1: Convergence theorem

The series solution Eq. (3) of problem (1) using LWM converges towards $u(x)$.

Proof:

Let $L^2(R)$ be the Hilbert space and let

$\psi_{k,n}(t)=|a|^{-\frac{1}{2}}\,\psi\left(a_0^{\,k}t-nb_0\right)$

where $\psi_{k,n}(t)$ form a basis of $L^2(R)$. In particular, when $a_0=2$ and $b_0=1$, $\psi_{k,n}(t)$ forms an orthonormal basis.

Let $u(x)=\sum_{i=1}^{M-1}C_{1i}\,\psi_{1i}(x)$ where $C_{1i}=\left\langle u(x),\psi_{1i}(x)\right\rangle$ for $k=1$ and $\langle .,.\rangle$ represents an inner product.

$$u(x)=\sum_{i=1}^{n}\left\langle u(x),\psi_{1i}(x)\right\rangle\,\psi_{1i}(x)$$

Let us denote $\psi_{1i}(x)$ as $\psi(x)$.

Let $\alpha_j=\left\langle u(x),\psi(x)\right\rangle$

Define the sequence of partial sums $\{S_n\}$ of $\left(\alpha_j\,\psi\left(x_j\right)\right)$; let S_n and S_m be arbitrary partial sums with $n\geq m$. We are going to prove that $\{S_n\}$ is a Cauchy sequence in Hilbert space.

Let $S_n=\sum_{j=1}^{n}\alpha_j\,\psi\left(x_j\right)$

$$\begin{aligned}\left\langle u(x),S_n\right\rangle&=\left\langle u(x),\sum_{j-1}^{n}\alpha_j\,\psi\left(x_j\right)\right\rangle\\&=\sum_{j=1}^{n}\overline{\alpha_j}\,\left\langle u(x),\psi\left(x_j\right)\right\rangle\\&=\sum_{j-1}^{n}\overline{\alpha_j}\,\alpha_j\\&=\sum_{j-1}^{n}\left|\alpha_j\right|^2\end{aligned}$$

We will claim that $\left\|S_n-S_m\right\|^2=\sum_{j=m+1}^{n}\left|\alpha_j\right|^2$ for $n>m$

$$\begin{aligned}\text{Now}\left\|\sum_{j=m+1}^{n}\alpha_j\,\psi\left(x_j\right)\right\|^2&=\left\langle\sum_{i=m+1}^{n}\alpha_i\,\psi\left(x_i\right),\sum_{j=m+1}^{n}\alpha_j\,\psi\left(x_j\right)\right\rangle\\&=\sum_{i=m+1}^{n}\sum_{j=m+1}^{n}\alpha_i\,\overline{\alpha_j}\,\left\langle\psi\left(x_i\right),\psi\left(x_j\right)\right\rangle\\&=\sum_{j=m+1}^{n}\alpha_j\,\overline{\alpha_j}\\&=\sum_{j=m+1}^{n}\left|\alpha_j\right|^2\end{aligned}$$

i.e. $\|S_n - S_m\|^2 = \sum_{j=m+1}^{n} |\alpha_j|^2$ for n > m.

From Bessel's inequality, we have $\sum_{j=1}^{\infty} |\alpha_j|^2$ is convergent and hence

$$\|S_n - S_m\|^2 \to 0 \quad as\ m,n \to \infty.$$

i.e. $\|S_n - S_m\| \to 0$ and $\{S_n\}$ is a Cauchy sequence and it converges to say 's'.

We assert that $u(x) = s$

Infact, $\left\langle S-u(x), \psi(x_j) \right\rangle = \left\langle S, \psi(x_j) \right\rangle - \left\langle u(x), \psi(x_j) \right\rangle$

$$= \left\langle \underset{n\to\infty}{Lt}\, S_n, \psi(x_j) \right\rangle - \alpha_j$$

$$= \underset{n\to\infty}{Lt} \left\langle S_n, \psi(x_j) \right\rangle - \alpha_j$$

$$= \alpha_j - \alpha_j$$

$\Rightarrow \left\langle S-u(x), \psi(x_j) \right\rangle = 0$

Hence $u(x)=s$ and $\sum_{j=1}^{n} \alpha_j \psi(x_j)$ converges to $u(x)$ and this completes the proof.

5. Error Estimation

In this part, error estimation for the approximate solution of Eq. (6) is discussed. Let us consider $e_n(x) = u(x) - \bar{u}(x)$ as the error function of the approximate solution $\bar{u}(x)$ for $u(x)$, where $u(x)$ is the exact solution of Eq. (6).

$\bar{u}(x) = G(x) + \int_0^x F(t, u(t))\,dt + H_n(x)$ where $H_n(x)$ is the perturbation term.

$$H_n(x) = \bar{u}(x) - G(x) - \int_0^x F(t, u(t))\,dt. \qquad (11)$$

We proceed to find an approximation $\overline{e_n}(x)$ to the error function $e_n(x)$ in the same way as we did before for the solution of the problem. Subtracting Eq. (11) from Eq. (6), the error function $e_n(t)$ satisfies the problem.

$$e_n(x) + \int_0^x F(t, u(t))\,dt = -\mathrm{H}_n(x) \qquad (12)$$

It should be noted that in order to construct the approximate $\overline{e_n}(x)$ to $e_n(x)$, only Eq. (12) needs to be recalculated in the same way as we did before for the solution of Eq. (6).

We ensure the stability of LWM through this convergence and error estimation.

6. Illustrative Examples

In this section, we apply LWM to solve some nonlinear IDE [4, 11, 17] and compare the LWM solutions of these problems with Homotopy perturbation Method (HPM), Wavelet Galerkin Method (WGM) and Hybrid methods.

Example 4.1

Consider the following nonlinear Volterra IDE:

$$u'(x) = 1 + \int_0^x u'(t)u(t)\,dt \qquad (13)$$

for $x \in [0,1]$ with the initial condition $u(0)=0$.

Applying LWM on Eq. (13), we have

$$C^T \psi(x_i) \approx x_i + \frac{x_i}{2}\sum_{j=1}^{s} w_j F\left(\frac{x_i}{2}(\tau+1), C^T \psi\left(\frac{x_i}{2}(\tau+1)\right)\right),$$
$$i = 1, 2, \ldots, 2^{k-1}M, \qquad (14)$$

where $F(x, u(x)) = \int_0^x u'(t)u(t)\,dx$

On solving equation Eq. (14) with k=1 and M=8, we find

$$c_{10} = 0,\ c_{11} = \frac{1}{2\sqrt{3}},\ c_{12} = 0,\ c_{13} = \frac{1}{120\sqrt{7}},\ c_{14} = 0,$$
$$c_{15} = \frac{1}{120960\sqrt{11}},\ c_{16} = 0, \ldots$$

Hence the approximate solution is

$u(x) = x + \frac{x^3}{6} + \frac{x^5}{30} + \frac{17}{2520}x^7 + \ldots$, and the exact solution is $\sqrt{2}\tan\left(\frac{x}{\sqrt{2}}\right)$.

We compare the LWM solution with the results given in [4, 11] and is presented in Table 1. The comparison between LWM and the exact solutions are depicted in Figure 1.

Figure 2. shows the plot of error using LWM with M = 2, 4, 8, and 16. From the Figure, we observe that LWM converges to the exact solution as M values increases.

Example 4.2

Consider the nonlinear Volterra IDE:

$$u'(x) = -1 + \int_0^x u^2(t)\,dt \qquad (15)$$

for $x \in [0,1]$ with the initial condition $u(0)=0$, Apply LWM on (15), we have

$$C^T \psi(x_i) \approx -x_i + \frac{x_i}{2}\sum_{j=1}^{s} w_j F\left(\frac{x_i}{2}(\tau+1), C^T \psi\left(\frac{x_i}{2}(\tau+1)\right)\right),$$
$$i = 1, 2, \ldots, 2^{k-1}M, \qquad (16)$$

where $F(x, u(x)) = \int_0^x u^2(t)\,dx$

On solving equation (16) with $k = 1$ and $M = 14$, we have

$$u(x) = -x + \frac{x^4}{12} - \frac{x^7}{252} + \frac{x^{10}}{6048} - \frac{x^{13}}{145152} + \dots$$

We compared the LWM solution with the results given in [4, 11] and is presented in Table 2.

Table 1. Numerical results of Example 4.1

X	Exact	WGM	HPM	LWM	Error
0	0	0	0	0	0
0.0625	0.06254	0.0626	0.06254	0.06253	1.0721e-05
0.1250	0.12532	0.1253	0.12533	0.12532	6.5413e-06
0.1875	0.18860	0.1886	0.18861	0.18860	6.4129e-06
0.2500	0.25263	0.2527	0.25264	0.25263	7.1357e-06
0.3125	0.31768	0.3177	0.31769	0.31768	7.6070e-06
0.3750	0.38404	0.3841	0.38403	0.38403	1.3496e-05
0.4375	0.45201	0.4520	0.45199	0.45198	3.2517e-05
0.5000	0.52193	0.5220	0.52188	0.52187	6.0515e-05
0.5625	0.59416	0.5942	0.59405	0.59404	1.2863e-04
0.6250	0.66914	0.6692	0.66890	0.66890	2.4193e-04
0.6875	0.74731	0.7473	0.74685	0.74684	4.7980e-04
0.7500	0.82923	0.8293	0.82838	0.82837	8.6897e-04
0.8125	0.91554	0.9156	0.91401	0.91400	1.5480e-03
0.8750	1.00688	1.0069	1.00433	1.00432	2.5661e-03
0.9375	1.10419	1.1042	1.10002	1.10002	4.1731e-03
1.0000	1.20846	1.2085	1.20185	1.20184	6.6202e-03

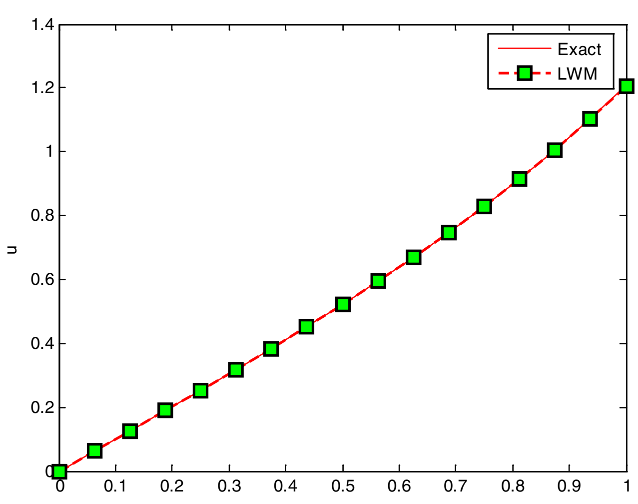

Figure 1. The comparison of the LWM solution and Exact solution of Example 4.1.

Example 4.3

Consider the following nonlinear Volterra IDE [17]

$$u'(x) - \int_0^x \cos(x-t)\, u^2(t)\, dt = -2\sin x - \frac{1}{3}\cos x - \frac{2}{3}\cos 2x \tag{17}$$

with the initial condition $u(0)=1$.

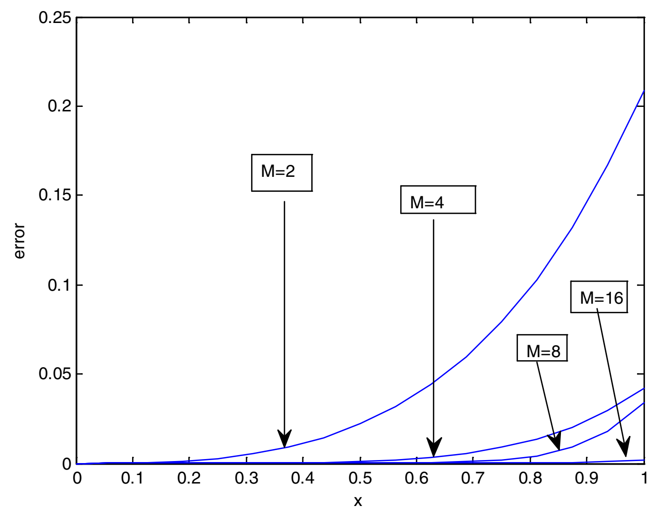

Figure 2. Plot of error for LWM with different values of M (2, 4, 8, 16).

Table 2. Numerical results of Example 4.2

X	WGM	HPM	LWM
0	0	0	0
0.0625	−0.0625	−0.06250	−0.06249
0.1250	−0.1250	−0.12498	−0.12497
0.1875	−0.1874	−0.18740	−0.18739
0.2500	−0.2497	−0.24968	−0.24967
0.3125	−0.3117	−0.31171	−0.31170
0.3750	−0.3734	−0.37336	−0.37335
0.4375	−0.4345	−0.43446	−0.43445
0.5000	−0.4948	−0.49482	−0.49482
0.5625	−0.5542	−0.55423	−0.55422
0.6250	−0.6124	−0.61243	−0.61243
0.6875	−0.6692	−0.66917	−0.66916
0.7500	−0.7242	−0.72415	−0.72415
0.8125	−0.7771	−0.77710	−0.77709
0.8750	−0.8277	−0.82767	−0.82766
0.9375	−0.8756	−0.87557	−0.87556
1.0000	−0.9205	−0.92048	−0.92047

Solving Eq. (17) with $k = 1$ and $M = 5$ using LWM, we get the approximate solution $u(x)=1-x-\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\ldots,$ and the exact solution is $u(x)=\cos x-\sin x.$

The comparison among the LWM solution besides the solution in [17] and exact solutions are shown in Table 3.

Figure 3 shows the comparison between LWM and the exact solutions of example 4.3.

Figure 4 shows the plot of error using LWM with $M = 2$, 4, and 8. From the Figure, we observe that LWM converges quickly to the exact solution even when $M = 8$.

Table 3. Numerical results of Example 4.3

X	Solution in [25]	LWM solution	Exact solution	Error
0.0	0.999999	1	1	0.0
0.1	0.895186	0.89517074	0.895170748	8e-09
0.2	0.781653	0.78139724	0.781397247	7e-09
0.3	0.659732	0.65981623	0.659816282	5.2e-08
0.4	0.530699	0.53164231	0.531642651	3.41e-07
0.5	0.398169	0.39815538	0.398157023	1.64e-06
0.6	0.260969	0.26068720	0.260693141	5.94e-06
0.7	0.120671	0.12060684	0.1206245	1.76e-05
0.8	−0.020638	-0.02069475	-0.020649381	4.53e-05
0.9	−0.161638	-0.16182136	-0.161716941	1.04e-04
1.0	−0.301983	-0.30138888	-0.301168678	2.20e-04

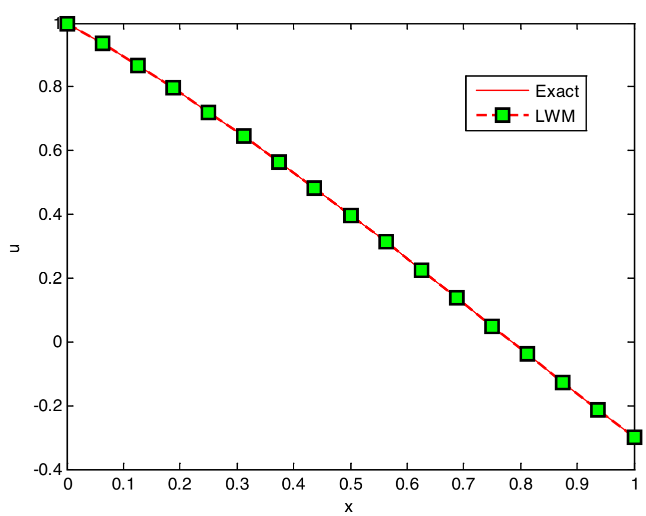

Figure 3. The comparison of LWM solution and Exact solution of Example 4.3.

Example 4.4

Consider the following nonlinear Volterra IDE [17]

$$u'(x)+u(x)-2\int_0^x \sin x\; u^2(t)\,dt = \cos x+(1-x)\sin x+\cos x\sin^2 x. \tag{18}$$

with the initial condition $u(0)=0$.

Solving Eq. (18) with $k=1$ and $M=8$ using LWM, we get the approximate solution $u(x)=x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{1}{5040}x^7+\ldots,$ and the exact solution is $u(x)=\sin x.$

The comparison among the LWM solution besides the solution in [17] and exact solutions are shown in Table 4. From Tables and Figures, we observe that the LWM solutions coincide with other mentioned methods.

7. Conclusion

In this work, we have proposed the Legendre wavelet method (LWM) for solving the nonlinear IDE. The properties of the Legendre wavelets together with the Gaussian integration method are used to reduce the problem to the solution of nonlinear algebraic equations. Also the convergence of the Legendre wavelet method is proved for the given function approximation. Illustrative examples clearly depict the validity and applicability of the technique.

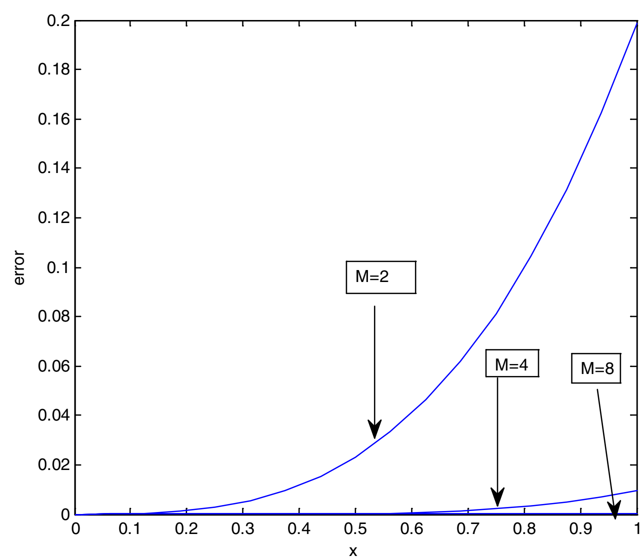

Figure 4. Plot of error for LWM with different values of M (2, 4, 8).

Table 4. Numerical results of Example 4.4

X	Solution in [25]	LWM solution	Exact solution	Error
0.0	0.000032	0.0	0.0	0.0
0.1	0.099825	0.099833416	0.09983341	6e-09
0.2	0.198678	0.198669333	0.19866933	3e-09
0.3	0.295603	0.29552025	0.29552020	5e-08
0.4	0.389605	0.389418666	0.38941834	3.26e-07
0.5	0.479398	0.479427083	0.47942553	1.55e-06
0.6	0.563598	0.564648	0.56464247	5.53e-06
0.7	0.642606	0.644233916	0.64421768	1.62e-05
0.8	0.715049	0.717397333	0.71735609	4.12e-05
0.9	0.779882	0.78342075	0.78332690	9.38e-05
1.0	0.837683	0.841666666	0.84147098	1.95e-04

Furthermore, since the basis of Legendre wavelets are polynomial, the values of integrals for the nonlinear integral equations of the form in Eq. (6) are calculated as approximately close to the exact solutions.

8. References

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