Indian Journal of Science and Technology, *Vol 7(3), 271–275, March 2014*
ISSN (Print): 0974-6846
ISSN (Online) : 0974-5645

Numerical Solution of Backward Stochastic Differential Equations Driven by Brownian Motion through Block Pulse Functions

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Abstract

In this paper, a computational technique is presented for solving a Backward Stochastic Differential Equations (BSDEs) driven by a standard Brownian motion. The proposed method is stated via a stochastic operational matrix based on the Block Pulse Functions (BPFs) in combination with the collocation method. With using this approach, the BSDEs are reduced to a stochastic nonlinear system of 2m equations and 2m unknowns. Then, the error analysis is proved by using some definitions, theorems and assumptions on the BSDEs. Efficiency of this method and good reasonable degree of accuracy is confirmed by some numerical examples.

Keywords: Backward Stochastic Differential Equations, Block Pulse Function, Brownian Motion, Stochastic Operational Matrix

1. Introduction

The BSDEs was first introduced by Pardoux and Peng[8]. In the last years these equations have been intensively investigated, the main reason for this investigations due to many application in mathematical finance, biology, medical, social, etc.[1]

In this paper we consider

$$\begin{cases} dX(s) = b(s, X(s), z(s, X(s)))ds + z(s, X(s))dB(s), \\ \qquad s \in (0,T), \\ X(T) = \xi, \end{cases} \tag{1}$$

or,

$$X(t) = \xi + \int_t^T b(s, X(s), z(s, X(s)))ds + \int_t^T z(s, X(s))dB(s), \quad t, s \in (0,T), \tag{2}$$

where, $X(t), b(s, X(s), z(s, X(s)))$ and $z(s, X(s))$ are the stochastic processes on the probability space (Ω, P) with the natural filtration $\digamma$ of B and unknown. Also, let $B(s)$ be the standard Brownian motion and $b:(0,T)\times R\times R \to R$, $z:(0,T)\times R \to R$.

Note that, in many fields of science there are number of problems that are dependent of the Eq. (2) and have many applications in physics, economics and biology[1]. Also, there are some numerical methods for Eq. (2) that can be classified into main groups: solving BSDEs by using the cubature method, carlo method, euler method[2,3,7], but we use from stochastic operational matrix based on properties of the BPFs and collocation method, because Eq. (2) is reduced to a stochastic nonlinear system of 2m equations and 2m unknowns with good approximate. Also, the method is easier than other methods.

The result of this paper is organized as follows: In Section 2, some definitions and theorems and assumptions on the coefficient of Eq. (2) is stateed. Also, the essential properties of the Block Pulse Functions (BPFs) are reviewed. In Section 3, Eq. (2) is reduced to a stochastic nonlinear system by using the stochastic operational matrix based on properties of the BPFs and collocation method. Error analysis is worked out in Section 4. In Section 5, the some numerical examples demonstrate applicability and accuracy of this method. Finally, in Section 6, is given a brief conclusion.

2. Preliminaries

Firstly, we state the following assumptions on the coefficient of Eq. (2) and definitions and theorems that are essential for this paper, then we introduce necessary properties the BPFs.

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Let the functions $b(t, X(t), z(t, X(t)))$ and $z(t, X(t))$ have Lipschitz conditions, i.e. there are constants L_1 and L_2 such that:

H1. $|b(t, x_1, y_1) - b(t, x_2, y_2)| < L_1(|x_1 - x_2| + |y_1 - y_2|)$.

H2. $|z(t, x) - z(t, y)| < L_2 |x - y|$.

For all $t \in (0, T)$, $T < 1$.

Definition 2.1: *Let $\nu = \nu(s, T)$ be the class of functions $f(t, \omega) : [0, \infty] \times \Omega \to R^n$ such that:*

The function $(t, \omega) \to f(t, \omega)$ is $\beta \times \digamma$ measurable and

1. The function f is adapted to $\digamma_t$.
2. β is the Borel algebra.
3. $E(\int_s^T f(t, \omega)^2 dt) < \infty$.

Theorem 2.2: *Let $f \in \nu(s, T)$ then*

$$E(\int_a^b f(t) dB(t))^2 = \int_a^b E(f^2(s)) ds.$$

Proof: To see the proof, refer[1].

Theorem 2.3: *If $B(t)$ be a Brownian motion on $[0, T]$ and $f(x)$ be twice continuously differentiable function on* R, *then for any $t \le T$*

$$df(B(t)) = f'(B(t)) dB(t) + \frac{1}{2} f''(B(t)) dt.$$

Proof: To see the proof, refer[1].

Finally, we state the main properties of the BPFs which are necessary for this paper. For more details, see[4–6].

Let

$$f(t) \approx \hat{f}_m(t) = \sum_{i=1}^{m} f_i \Phi_i(t),$$

where,

$$\Phi_i(t) = \begin{cases} 1 & (i-1)h \le t < ih, \quad i = 1, \ldots, m, \\ 0 & otherwise, \end{cases}$$

with $h = \frac{T}{m}$ and $f_i = \frac{1}{h} \int_0^T f(t) \Phi_i(t) dt$.

In the vector form we have,

$$f(t) \approx \hat{f}_m(t) = F^T \Phi(t) = \Phi^T(t) F,$$

where,

$$F = (f_1, f_2, \ldots, f_m)^T,$$

and

$$\Phi(t) = (\Phi_1(t), \Phi_2(t), \cdots, \Phi_m(t))^T, t \in [0, T).$$

Also, we have

$$\int_0^t \Phi(s) ds = P \Phi(t),$$

where,

$$P = \frac{h}{2} \begin{pmatrix} 1 & 2 & 2 & \ldots & 2 \\ 0 & 1 & 2 & \ldots & 2 \\ 0 & 0 & 1 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix}_{m \times m}$$

3. Solving Backward Stochastic Differential Equations

Theorem 3.1: *Let $\Phi(t) = (\Phi_1(t), \Phi_2(t), \ldots, \Phi_i(t), \ldots, \Phi_m(t))^T$ where, $\Phi_i(t)$, i=1,2,…,m denotes the BPFs, then*

$$\int_t^0 \Phi(s) dB(s) \approx -P_s \Phi(t), \qquad t \in [0, T),$$

where,

$$P_s = \begin{pmatrix} B(\frac{h}{2}) & B(h) & B(h) & \ldots & B(h) \\ 0 & B(\frac{3h}{2}) - B(h) & B(2h) - B(h) & \ldots & B(2h) - B(h) \\ 0 & 0 & B(\frac{5h}{2}) - B(2h) & \ldots & B(3h) - B(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & B(\frac{(2m-1)h}{2}) - B((m-1)h) \end{pmatrix}_{m \times m}$$

Proof: By using defined $\Phi_i(t), i = 1, 2, \ldots, m$, we get

B1. If $t \in [0, (i-1)h)$, then

$$\int_t^0 \Phi_i(s) dB(s) = 0$$

B2. If $t \in [(i-1)h, ih)$, then

$$\int_t^0 \Phi_i(s) dB(s) = \int_t^{(i-1)h} \Phi_i(s) dB(s) + \int_{(i-1)h}^0 \Phi_i(s) dB(s) = B((i-1)h) - B(t) = -(B(t) - B((i-1)h)).$$

B3. If $t \in [ih, T)$, then

$$\int_t^0 \Phi_i(s) dB(s) = \int_t^{ih} \Phi_i(s) dB(s) + \int_{ih}^{(i-1)h} \Phi_i(s) dB(s) + \int_{(i-1)h}^0 \Phi_i(s) dB(s) = B((i-1)h) - B(ih) = -(B(ih) - B((i-1)h)).$$

From **B1**, **B2** and **B3**, we can write

$$\int_t^0 \Phi_i(s) dB(s) = -\begin{cases} 0 & 0 \le t < (i-1)h, \\ B(t) - B((i-1)h) & (i-1)h \le t < ih, \\ B(ih) - B((i-1)h) & ih \le t < T. \end{cases} \tag{3}$$

Let

$$B(t)-B((i-1)h)\approx B((i-0.5)h)-B((i-1)h),\ t\in[(i-1)h,ih),$$

so, the relation (3) is equality with

$$\int_t^0 \Phi_i(s)dB(s)\approx -\begin{cases}0 & 0\le t<(i-1)h,\\ B((i-0.5)h)-B((i-1)h) & (i-1)h\le t<ih,\\ B(ih)-B((i-1)h) & ih\le t<T,\end{cases} \tag{4}$$

by using the relation (4), we get

$$\int_t^0 \Phi_i(s)dB(s)\approx -(0,\ldots,0,B((i-0.5)h)-B((i-1)h),\\ B(ih)-B((i-1)h),\ldots,B(ih)-\\ B((i-1)h))\Phi(t). \tag{5}$$

Now, by using the relation (5), we get

$$\int_t^0 \Phi(s)dB(s)\approx -P_s\Phi(t),$$

where

$$P_s=\begin{pmatrix} B(\frac{h}{2}) & B(h) & B(h) & \ldots & B(h)\\ 0 & B(\frac{3h}{2})-B(h) & B(2h)-B(h) & \ldots & B(2h)-B(h)\\ 0 & 0 & B(\frac{5h}{2})-B(2h) & \ldots & B(3h)-B(2h)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & B(\frac{(2m-1)h}{2})-B((m-1)h)\end{pmatrix}_{m\times m}$$

Let

$$\begin{cases} b(s,X(s),z(s,X(s)))=f(s),\\ z(s,X(s))=g(s),\end{cases} \tag{6}$$

by substituting the relation (6) in Eq. (2), we obtain

$$X(t)=\xi+\int_t^T f(s)ds+\int_t^T g(s)dB(s). \tag{7}$$

Now, we suppose

$$m=T-s, \tag{8}$$

by using the relation (8) and Theorem (2.3), we get

$$X(t)=\xi-\int_{T-t}^0 f(m)dm-\int_{T-t}^0 g(m)dB(m). \tag{9}$$

By using properties of the BPFs, we can write

$$\begin{cases} f(m)\approx f^T\Phi(m)=\Phi^T(m)f,\\ g(m)\approx g^T\Phi(m)=\Phi^T(m)g,\end{cases} \tag{10}$$

by using the relation (10), we get

$$X(t)\approx \xi-\int_{T-t}^0 f^T\Phi(m)dm-\int_{T-t}^0 g^T\Phi(m)dB(m), \tag{11}$$

or,

$$X(t)\approx \xi+f^TP\Phi(T-t)+g^TP_s\Phi(T-t). \tag{12}$$

Now, with replacing $\approx$ by $=$, substituting the relation (12) into (6) and collocation method in m nodes $T-t_j=\dfrac{j}{\frac{1}{T}m+1}$, $j=1,\ldots,m$, we get

$$\begin{cases} f(t_j)=b(t_j,\xi+f^TP\Phi(T-t_j)+g^TP_s\Phi(T-t_j),\\ z(t_j,\xi+f^TP\Phi(T-t_j)+g^TP_s\Phi(T-t_j))),\\ g(t_j)=z(t_j,\xi+f^TP\Phi(T-t_j)+g^TP_s\Phi(T-t_j)),\end{cases} \tag{13}$$

or,

$$\begin{cases} f^T\Phi(t_j)=f(t_j)=b(t_j,\xi+f^TP\Phi(T-t_j)+\\ g^TP_s\Phi(T-t_j),z(t_j,\xi+f^TP\Phi(T-t_j)+\\ g^TP_s\Phi(T-t_j))),g^T\Phi(t_j)=g(t_j)=z(t_j,\xi+\\ f^TP\Phi(T-t_j)+g^TP_s\Phi(T-t_j)).\end{cases} \tag{14}$$

After solving Eq. (14) where, the stochastic nonlinear system of 2m equations and 2m unknowns, we can write

$$X(t)=X_m(t)=\xi+f^TP\Phi(T-t)+g^TP_s\Phi(T-t). \tag{15}$$

4. Error Analysis

Theorem 4.1: *Let $f(t)$ be an arbitrary real bounded function on interval $[0,1)$, and $e(t)=f(t)-\hat{f}_m(t)$, $t\in[0,1)$, that $\hat{f}_m(t)$, is the block pulse approximation of $f(t)$. Then*

$$|e(t)|^2\le O(h^2).$$

Proof: To see the proof, refer[6].

Let

$$\begin{cases} \hat{f}(s)=\hat{b}(s,X_m(s),\hat{z}(s,X_m(s))),\\ \hat{g}(s)=\hat{z}(s,X_m(s)),\end{cases}$$

where $\hat{f}(t)$ and $\hat{g}(s)$ are approximated by properties of the BPFs. Also, we define $e_m(t)=X(t)-X_m(t)$ that $X_m(t)$ is defined in Eq. (15) and $X(t)$ be solution of Eq. (2). Now, we can write

$$e_m(t) = X(t) - X_m(t) = \int_0^{T-t} (f(s) - \hat{f}(s))ds + \int_0^{T-t} (g(s) - \hat{g}(s))dB(s). \quad (16)$$

Also, we define

$$\begin{cases} f_m(s) = b(s, X_m(s), z(s, X_m(s))), \\ g_m(s) = z(s, X_m(s)). \end{cases}$$

Theorem 4.2 *Let conditions* H1, H2 *hold,* $X_m(t)$ is defined in Eq. (15) and be solution of Eq. (2). *Then*

$$\| X(t) - X_m(t) \|^2 \le O(h^2), \ t \in (0,T), T < 1,$$

where $\| X \|^2 = E[X^2]$.

Proof.

$$e_m(t) = X(t) - X_m(t) = \int_0^{T-t} (f(s) - \hat{f}(s))ds + \int_0^{T-t} (g(s) - \hat{g}(s))dB(s)),$$

from Theorem (2.2) and $(x + y)^2 \le 2(x^2 + y^2)$, *we can write*

$$\|x(t) - x_m(t)\|^2 \le \left\| \int_0^{T-t} (f(s) - \hat{f}(s))ds + \int_0^{T-t} (g(s) - \hat{g}(s)) \, dB(s) \right\|^2$$

$$\le 2(\| \int_0^{T-t} (f(s) - \hat{f}(s))ds \|^2 + \| \int_0^{T-t} (g(s) - \hat{g}(s))dB(s) \|^2), \quad (17)$$

$$\le 2(\int_0^{T-t} \| (f(s) - \hat{f}(s)) \|^2 \, ds + \int_0^{T-t} \| (g(s) - \hat{g}(s)) \|^2 \, ds)$$

$$\le 4(\int_0^{T-t} \| (f(s) - f_m(s)) \|^2 \, ds + \int_0^{T-t} \| (f_m(s) - \hat{f}(s)) \|^2 \, ds + \int_0^{T-t} \| (g(s) - g_m(s)) \|^2 \, ds + \int_0^{T-t} \| (g_m(s) - \hat{g}(s)) \|^2 \, ds). \quad (18)$$

By using Theorem (4.1), we get

$$\begin{cases} \| \hat{f}(s) - f_m(s) \|^2 \le k_1 h^2, \\ \| \hat{g}(s) - g_m(s) \|^2 \le k_2 h^2. \end{cases} \quad (19)$$

By using Lipschitz conditions, we obtain

$$\begin{cases} \| f(s) - f_m(s) \|^2 \le L_1 (\| X(s) - X_m(s) \|^2 + \\ \| z(s, X(s)) - z(s, X_m(s)) \|^2), \\ \| g(s) - g_m(s) \|^2 \le L_2 \| X(s) - X_m(s) \|^2, \end{cases} \quad (20)$$

or,

$$\begin{cases} \| f(s) - f_m(s) \|^2 \le L_1 (\| X(s) - X_m(s) \|^2 + L_2 \| X(s) - X_m(s) \|^2), \\ \| g(s) - g_m(s) \|^2 \le L_2 \| X(s) - X_m(s) \|^2 \end{cases} \quad (21)$$

Now, by substituting the relations (19) and (21) in the relation (18), we get

$$\| x(t) - x_m(t) \|^2 \le 4[L_1 (\int_0^{T-t} \| X(s) - X_m(s) \|^2 \, ds + L_2 \int_0^{T-t} \| X(s) - X_m(s) \|^2 \, ds) + k_1 h^2 + L_2 \int_0^{T-t} \| X(s) - X_m(s) \|^2 \, ds + k_2 h^2], \quad (22)$$

if we define $p = 4k_1 h^2 + 4k_2 h^2, q = 4L_1 + 4L_1 L_2 + 4L_2$ and $U(t) = \| X(t) - X_m(t) \|^2$, then we obtain

$$U(t) \le p + q \int_0^{T-t} U(s)ds, \quad (23)$$

so, by Gronwall inequality, we can write

$$U(t) \le p(1 + q \int_0^{T-t} e^{q(T-t-s)} ds), \quad t \in (0,T), \ T > 0,$$

or,

$$\| X(t) - X_m(t) \|^2 \le O(h^2), \ t \in (0,T), \ T < 1$$

5. Numerical Examples

Example 1: *Let us consider the BSDE*

$$\begin{cases} dX(s) = \dfrac{1 - X(s)}{1 - s} ds + dB(s), s \in (0,1), \\ X(1) = 1, \end{cases} \quad (24)$$

with exact solution $X(t) = t + (1-t) \int_0^t \frac{dB(s)}{1-s}$. The numerical results have been shown in Table (1), where $\bar{x}_E$ and s_E are error mean and standard deviation of error, respectively.

Example 2: *Let us consider the BSDE*

$$\begin{cases} dX(s) = (\dfrac{2}{1-4s} + \dfrac{-4X(s)}{1-4s})ds + dB(s), \ s \in (0, 0.25), \\ X(0.25) = \dfrac{1}{2}, \end{cases} \quad (25)$$

with exact solution $X(t) = 2t + (1-4t) \int_0^t \frac{1}{1-4s} dB(s)$.

The numerical results have been shown in Table (2), where $\bar{x}_E$ and s_E are error mean and standard deviation of error, respectively.

6. Conclusion

In this paper, a numerical method to solve the BSDEs driven by the Brownian motion is presented. The proposed method is introduced via the stochastic operational matrix based on the properties of the BPFs and collocation

Table 1. Mean, Standard deviation and Confidence interval for error mean (m=8 and T=1)

t	$\bar{x}_E$	S_E	%95 confidence interval for mean of E	
			Lower	Upper
0.2	7.98820×10^{-1}	4.66082×10^{-3}	7.95931×10^{-1}	8.01709×10^{-1}
0.4	5.99573×10^{-1}	4.84417×10^{-3}	5.96571×10^{-1}	6.02575×10^{-1}
0.6	3.99935×10^{-1}	4.68170×10^{-3}	3.97033×10^{-1}	4.02837×10^{-1}
0.8	2.00222×10^{-1}	4.58602×10^{-3}	1.97380×10^{-1}	2.03064×10^{-1}

Table 2. Mean, Standard deviation and Confidence interval for error mean (m=8 and T=0.25)

t	$\bar{x}_E$	S_E	%95 confidence interval for mean of E	
			Lower	Upper
0.05	4.42961×10^{-1}	1.5400898×10^{-2}	4.3341542×10^{-1}	4.5250657×10^{-1}
0.1	3.932480×10^{-1}	1.4719858×10^{-2}	$3.84124537 \times 10^{-1}$	$4.02371462 \times 10^{-1}$
0.15	3.4351100×10^{-1}	$1.40863312 \times 10^{-2}$	3.3478020×10^{-1}	3.5224179×10^{-1}
0.2	2.9371900×10^{-1}	1.3537731×10^{-2}	2.8532822×10^{-1}	3.0210977×10^{-1}

method. Also, In this method Eq. (2) is reduced to the stochastic nonlinear system with good approximate that is easier than other numerical methods. Note that the important purpose of this paper is to compare between the numerical solution and the exact solution of BSDEs, because there are a few the exact solution for Eq. (2). Also, the proposed method is evaluated by using some numerical examples.

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