Indian Journal of Science and Technology, *Vol 9(7), DOI: 10.17485/ijst/2016/v9i7/84188, February 2016*
ISSN (Print) : 0974-6846
ISSN (Online) : 0974-5645

Fixed Point Result for $\alpha P_{f,g}$–Integral Contractive Mappings with Applications

Mecheraoui Rachid

Department of Mathematics and Informatics, Abbes Laghrour University-Khenchela-Algeria;
rachid.mecheraoui@yahoo.fr

Abstract

Background/Objectives: In this paper, we introduced the concept of $\alpha P_{f,g}$ integral contractive mappings, which is a new class of integral contractive mappings and using this notion we establish a new fixed point theorem. **Findings:** Our paper represents a generalization and extension of fixed point theorems for mappings satisfying contractive conditions of integral type where the contractive inequality depends on rational and irrational expression. In particular, we omitted the condition of continuity (which is a very strong condition and appear in almost all papers using contractive mapping of rational type) from many existing results. **Application/Improvements:** As a direct consequence, some new results of integral type for rational and irrational contraction maps are presented to illustrate our obtained result.

Keywords: Fixed Point, Integral Type, $\alpha P_{f,g}$–Integral Contractive Mappings, Irrational Type, Rational Type

1. Introduction and Preliminaries

1. In 2002, Branciari[1] introduced the notion of contractive mappings of integral type in complete metric spaces. Afterwards, many researchers extended this result to more general contractive conditions of integral type providing sufficient assumptions which ensure the existence and uniqueness of fixed points (see for example Gupta and Saxena[2], Liu et al.[3], Pathak[4], Rhoades[5] and Vishal and Ashima[14] Vetro).

Throughout this paper, let us consider the following sets:

$$\Phi_1 = \left\{ \begin{array}{c} \varphi; \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is locally Lebesgue integrable function and} \\ \int_0^{\varepsilon} \varphi(t)dt > 0 \text{ for each } \varepsilon > 0 \end{array} \right\},$$

$$\Phi_2 = \left\{ \begin{array}{c} \psi; \psi : \mathbb{R}_+ \to \mathbb{R}_+ \text{ is a lower semi} - \text{continuous function with} \\ \psi(0) = 0, \psi(t) > 0 \text{ for each } t > 0 \text{ and } \lim_{t\to\infty} \psi(t) > 0 \end{array} \right\}.$$

In 2014, Liu and al.[6] extended a result established in the paper[7] by proving the following theorem:

Theorem 1.1. Let (φ, ψ) be in $\Phi_1 \times \Phi_2$, $M \in \{M_1, M_2, M_3, M_4\}$ and T be a mapping from a complete metric space (X, d) into itself satisfying

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le \int_0^{M(x,y)} \varphi(t)dt + \int_0^{\psi(M(x,y))} \varphi(t)dt.$$

Then T has a unique fixed point $a \in X$ such that $lim_{n\to\infty} T^n x = a$ for each $x \in X$.

Where

$$M_1(x,y) = max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(Tx,y) + d(x,Ty)]\right\},$$

$$M_2(x,y) = max\left\{d(x,Tx), d(y,Ty)\right\},$$

$$M_3(x,y) = max\left\{d(x,y), d(x,Tx), d(y,Ty)\right\},$$

$$M_4(x,y) = max\left\{d(x,Tx), d(y,Ty), \frac{1}{2}[d(Tx,y) + d(x,Ty)]\right\}.$$

In[8], Dass and Gupta proved a fixed point theorem by considering a class of mappings where the contractive inequality depends on rational expressions:

$$d(Tx,Ty) \le M_5(x,y) := \alpha d(y,Ty)\frac{1 + d(x,Tx)}{1 + d(x,y)} + \beta d(x,y),$$

A generalization of the above contraction was suggested by Gupta and Saxena[9] (among others, see in particular[10] and[11]) by introducing the following class of mappings:

*Author for correspondence

$$d(Tx,Ty)\le a_1 d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}+a_2\frac{d(x,Tx)d(y,Ty)}{d(x,y)}+a_3 d(x,y).$$

Another well-known theorem was established by Jaggi[12] which may be considered as an extension of the two last results. He considered the following class of mappings:

$$d(Tx,Ty)\le a_1\frac{d(x,Tx)d(y,Ty)}{d(x,y)}+a_2 d(x,y).$$

To prove the main results in[9] and[12], authors impose a very strong and unwanted condition: "T is continuous". These results have been generalized in many ways over the years but the condition of continuity appeared in all the subsequent papers using this type of contractions, as an essential condition.

In our paper, we introduce the concept of $\alpha P_{f,g}$ integral contractive mappings, which is a new class of integral contractive mappings and using this notion we establish a new fixed point theorem. This class of contractions extends and generalizes more or less known results including all previous results[6,8,9,12]. In particular, we omit the condition of continuity of the mapping "T" from the main result in[9,12] (see corollary 1). Moreover, by corollaries 2, 3, we introduce a new class of rational and irrational contractive mappings and give their related theorems.

Throughout this paper, we consider the following notations:

Notation 1.2. *Let* $f:\mathbb{R}_+^3\times\mathbb{R}^*\to\mathbb{R}_+$ *be a function.* $\prod_{f(x,y,z,t)}$ *denotes the set defined by:*

$$\prod_{f(x,y,z,t)}:=\{k;k\in\{x,y,z,t\}\ \text{and } k \text{ apear in the equation of } f(x,y,z,t)\}.$$

Example 1.3. *Let* $f:\mathbb{R}_+^3\times\mathbb{R}^*\to\mathbb{R}_+$ *be a function defined by:*

$$f(x,y,z,t)=x+t$$

then $\prod_{f(x,y,z,t)}=\{x,t\}$.

Notation 1.4. *We denote by* Υ *the set of all functions* $f:\mathbb{R}_+^3\times\mathbb{R}^*\to\mathbb{R}_+$ *satisfying the following assumptions:*

(H_1)

(H_{1a}) – *Function* $(x, y, z)\to f(x, x, y, z)$ *is non-decreasing in each of its variables.*

(H_{1b}) – *There exists* $k\in\{x,y\};\{z,k\}\subset\prod_{f_i(x,y,z,t)}$.

(H_2) *For all sequences* $\{x_n^1\}_n,\{x_n^2\}_n,\{x_n^3\}_n,\{x_n^4\}_n$ *in* $\mathbb{R}_{+,}$ *there exists* $\delta\in[0,1]$ *such that, for large values of n:*

(H_{2a}) $-f(X_n)\ge\delta\ min\Omega-\{0,x_n^4\}$ if $\mu=0$.

(H_{2b}) $-f(X_n)\ge\delta\ min\{\xi,x_n^i;x_n^i\in\prod_{f(X_n)}\wedge\xi=\xi_i\}$ *if one of the following conditions hold true:*

- $\mu\ne0,\xi=\xi_3$ *and* $\xi_1=\xi_2=0$.
- $\mu\ne0,\xi\in\{\xi_1,\xi_4\}$ *and* $x_n^2\le2x_n^1$.

(H_{2c}) $-f(X_n)\le max\ \Omega$ *if one of the following conditions hold true:*

- $\{x_n^2\le\min\{x_n^4,2x_n^1\}\}$ *and* {*there exists* $i\in\{1,3\}$ *such that* $\xi_i=\mu\ne0$ *and* $x_n^i\in\prod_{f(X_n)}\}$.
- $\xi_1=\cdots=\xi_4$.

Where

$$I_4=\{1,...,4\},\quad \Omega=\{x_n^1,...,x_n^4\},\quad X_n=(x_n^1,...,x_n^4),$$

$$\xi_i=\lim_{n\to\infty}\inf f\,x_n^i,\ \mu=max\{\xi_i;i\in I_4\},\ \xi=min\{\xi_i;i\in I_4\wedge x_n^i\in\prod_{f_i(X_n)}\}.$$

Definition 1.5. *Let* $f:\mathbb{R}_+^3\times\mathbb{R}^*\to\mathbb{R}_+$ *be a function.* If there *exist two functions g, h in* Υ *such that*

$$g(X)\le f(X)\le h(X)$$

for all $X\in\mathbb{R}_+^3\times\mathbb{R}^*$, *function f is called* Υ*– addmissible.*

Notation 1.6. *For all* $f:\mathbb{R}_+^3\times\mathbb{R}^*\to\mathbb{R}_+$, *we denote*

$$P_f(x,y):=f(d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(Tx,y)]).\quad(1.1)$$

Remark 1.7. Let us observethat the following mappings $g_i:\mathbb{R}_+^4\to\mathbb{R}_+$:

$$g_1(x,y,z,t)=max\{x,y,z,t\},\quad g_2=(x,y,z,t)=max\{y,z\},$$
$$g_3(x,y,z,t)=max\{x,y,z\},\quad g_4(x,y,z,t)=max\{y,z,t\},$$
$$g_5(x,y,z,t)=\alpha z\frac{1+y}{1+x}+\beta x,\quad g_6(x,y,z,t)=a[y+z],$$

satisfy assumptions (H_1)–(H_2) and for all i = 1, …, 6, we have

$$P_{g_i}(x,y)=g_i(d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)+\mathrm{d}(Tx,y)])$$
$$=M_i(x,y).$$

Notation 1.8. We denote by $\theta: \mathbb{R}_+ \to \mathbb{R}_+$ a function satisfying the following conditions
A_1 – θ is non-decreasing and the sequence $\{\theta_n(t)\}_n$ defined by $\theta_n(t) := \theta^n(t)$ (n^{th} iterate of θ) is bounded for all $t \in \mathbb{R}_+^*$.
A_2 – There exists $(\in, r) \in [0,1] \times [1, +\infty]$, for all $(t, \eta) \in \mathbb{R}_+^* \times [0,1] + \in [; \theta(\eta t) \leq \eta^r \theta(t)$.

Remark 1.9. As we can easily verify, condition (A_2) imply that

- for all $a \in \mathbb{R}_+$ we have $lim_{t \to a} \theta(t) \leq \theta(a)$.
- $\theta(0) = 0$.
- In particular, θ is continuous from the right at the point 0.

Remark 1.10. All (c)-comparison function satisfies conditions A_1 and A_2.

Notation 1.11. Let T be a mapping from a complete metric space (X, d) into itself, we denote by $\Gamma(T)$ the set of all functions $\alpha: X \times X \to \mathbb{R}_+$ satisfying the following conditions

1. $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$,
2. $\left.\begin{matrix} \alpha(x, y) \geq 1 \\ \alpha(y, z) \geq 1 \end{matrix}\right\} \Rightarrow \alpha(x, z) \geq 1$,
3. $lim_{n \to \infty} d(x_n, y_n) = 0 \Rightarrow liminf_{n \to \infty} \alpha(x_n, y_n) \geq 1$,

for all $x, y, z \in X$ and $\{x_n\}, \{y_n\}$ nonnegative sequences.

The following lemmas play an important role to obtain our result.

Lemma 1.12.[13] Let $\varphi \in \Phi_1$ and $\{r_n\}_n$ be a non-negative sequence with $lim_{n \to \infty} r_n = a$. Then

$$\lim_{n \to \infty} \int_0^{r_n} \varphi(t)dt = \int_0^a \varphi(t)dt.$$

Lemma 1.13. [13] Let $\varphi \in \Phi_1$ and $\{r_n\}_n$ be a nonnegative sequence. Then

$$\lim_{n \to \infty} \int_0^{r_n} \varphi(t)dt = 0,$$

if and only if $lim_{n \to \infty} r_n = 0$.

Lemma 1.14. Let $\varphi \in \Phi_1$ and $\{u_n\}_n$, $\{v_n\}_n$ be two nonnegative sequences such that $u_n \leq v_n$ for all $n \in \mathbb{N}$ and $lim_{n \to \infty} u_n = lim_{n \to \infty} v_n = l \in \mathbb{R}$, then

$$\lim_{n \to \infty} \int_{u_n}^{v_n} \varphi(t)dt = 0.$$

Proof. The condition $lim_{n \to \infty} u_n = lim_{n \to \infty} v_n = l \in \mathbb{R}$ with the fact that $\varphi \in \Phi_1$ means that

$$\forall n \in \mathbb{N}, \exists n_0 \geq n, \forall m \geq n_0; \int_{u_m}^{v_m} \varphi(t)dt \leq \int_{l-\frac{1}{m}}^{l+\frac{1}{m}} \varphi(t)dt$$
$$= \int_0^{l+\frac{1}{m}} \varphi(t)dt - \int_0^{l-\frac{1}{m}} \varphi(t)dt$$

This, with lemma 1.12, clearly drives to the result.

Lemma 1.15. Let $\varphi \in \Phi_1$ and $\{r_n\}_n$ be a nonnegative bounded sequence and $a, b \in \mathbb{R}_+^*$ such that

$$\int_0^{r_{n+1}} \varphi(t)dt \leq \theta\left(\int_0^{r_n} \varphi(t)dt\right) - a\int_0^b \varphi(t)dt,$$

for all $n \in \mathbb{N}$. Then

$$\exists N \in \mathbb{N} \mid \forall n \geq N;\ r_n = 0.$$

Proof Let $\varphi \in \Phi_1$ and $\{r_n\}_n$ be a nonnegative bounded sequence. Assume that there exist $a, b \in \mathbb{R}_+^*$, such that

$$\int_0^{r_{n+1}} \varphi(t)dt \leq \theta\left(\int_0^{r_n} \varphi(t)dt\right) - a\int_0^b \varphi(t)dt.$$

Taking into account condition A_1 and the fact that $\{r_n\}_n$ is bounded, we get

$$\int_0^{r_{n+1}} \varphi(t)dt \leq \theta\left(\theta\left(\int_0^{r_{n-1}} \varphi(t)dt\right)\left[1 - \left(a\int_0^b \varphi(t)dt \Big/ \theta\left(\int_0^{sup\{r_n\}} \varphi(t)dt\right)\right)\right]\right) - a\int_0^b \varphi(t)dt.$$

Without loss of generality, we can suppose that

$$\theta\left(\int_0^{r_n} \varphi(t)dt\right) \geq a\int_0^b \varphi(t)dt.$$

Then, putting $\vartheta_1 := 1 - \left(a\int_0^b \varphi(t)dt \Big/ \theta\left(\int_0^{sup\{r_n\}} \varphi(t)dt\right)\right)$ and using condition A_2, we obtain

$$\int_0^{r_{n+1}} \varphi(t)dt \leq \vartheta_1^r \theta^2\left(\int_0^{r_{n-1}} \varphi(t)dt\right) - a\int_0^b \varphi(t)dt.$$

Continuing in this way, we obtain

$$\int_0^{r_{n+1}} \varphi(t)dt \leq \vartheta_1^{\sum_{i=1}^{n-1} r^i} \theta^n\left(\int_0^{r_1} \varphi(t)dt\right) - a\int_0^b \varphi(t)dt.$$

From the fact that the sequence $\{\theta^n(t)\}_n$ is bounded for all t, $\vartheta_1 < 1$ and $r \geq 1$, we deduce that the right hand side of the last inequality tends to $-a\int_0^b \varphi(t)dt$ while $n \to \infty$, which is a clear contradiction with the fact that $a, b \in \mathbb{R}_+^*$ and $\varphi \in \Phi_1$.

We next define the concept of $aP_{f,g}$–integral contractive mappings:

Definition 1.16. Let (φ, ψ) be in $\Phi_1 \times \Phi_2$, $f, g: \mathbb{R}_+^4 \to \mathbb{R}_+$, $a: X \times X \to \mathbb{R}_+$ be a functions and T a mapping from a complete metric space (X, d) into itself satisfying the integral inequality:

$$a(x,y)\int_0^{d(Tx,Ty)} \varphi(t)dt \le \theta\left(\int_0^{P_f(x,y)} \varphi(t)dt\right) - \int_0^{\psi(Pg(x,y))} \varphi(t)dt, \quad (1.2)$$

for all x, y in X. Then T is called $aP_{f,g}$–integral contractive mapping.

2. Main Result

Theorem 2.1. Let T be an aP_{f_1,f_2} integral contractive mapping from a complete metric space (X, d) into itself, where a is in $\Gamma(T)$ and f_1, f_2 are two Υ–admissible functions. If $a(x_0, Tx_0) \ge 1$ for some $x_0 \in X$ then, T has a fixed point $a \in X$ and the sequence $\{T^n x_0\}_n$ converges to a. Moreover, if for all fixed points x and y of the mapping T, we have $a(x, y) \ge 1$, the mapping T has a unique fixed point.

Proof. First suppose -without loss of generality- that

$$x_n \ne x_{n+1} and\ a(x_n, x_m) \ge 1, \quad (2.1)$$

for any positive integers n,m, where $x_n := T^n x_0$. On the other hand, by the facts that f_1, f_2 are two Υ–admissible functions, they can be regarded as two elements of T without anyloss of generality.

Now, denoting by $d_n := d(x_n, x_{n+1})$, let's show that the sequence $\{d_n\}_n$ is decreasing to 0 after some rank. For this, assume that for all $n_0 \in \mathbb{N}$, there exists n $> n_0$ such that $d_n \le d_{n+1}$. The case $\{d_n\}_n$ is an unbounded sequence leads to an obvious contradiction. In fact, considera non-decreasing subsequence $\left\{d_{n_k}\right\}_k$ with the following properties:

- d_n tends to $+\infty$.
- $d_{n_k} > d_{n_k-1}$ for all $k \in \mathbb{N}$.

Invoking assumptions $(H_{1a,1b,2b})$ and relation (1.1) we obtain

$$\begin{aligned} P_{f_2}(x_{n_k-1}, x_{n_k}) &= f_2\left(d_{n_k-1}, d_{n_k-1}, d_{n_k}, \frac{1}{2}d(x_{n_k-1}, x_{n_k+1})\right) \\ &\ge f_2\left(0, 0, d_{n_k}, \frac{1}{2}(d_{n_k} - d_{n_k-1})\right) \\ &\ge \delta_1 d_{n_k}, \end{aligned}$$

which with the fact that $\psi \in \Phi_2$ means

$$lim_{k\to\infty}\psi\left(P_{f_2}(x_{n_k}, x_{n_k+1})\right) \in [0, +\infty].$$

Moreover, due to assumptions $(H_{1a,2c})$ and the first property of the sequence $\{d_{n_k}\}_k$ we have

$$\begin{aligned} P_{f_1}(x_{n_k-1}, x_{n_k}) &\le f_1\left(d_{n_k-1}, d_{n_k-1}, d_{n_k}, \frac{1}{2}(d_{n_k} + d_{n_k-1})\right) \\ &\le f_1(d_{n_k}, d_{n_k}, d_{n_k}, d_{n_k}) \\ &\le d_{n_k}. \end{aligned}$$

The last two results, with the integral inequality (1.2), relation (2.1) and condition A_1, imply that

$$\int_0^{d_{n_k}} \varphi(t)dt \le \theta\left(\int_0^{d_{n_k}} \varphi(t)dt\right) - \int_0^{min\left\{1, \frac{1}{2}\lim_{k\to\infty}\psi\left(P_{f_2}(x_{n_k}, x_{n_k+1})\right)\right\}} \varphi(t)dt,$$

which is in contradiction with Lemma 1.15. Now, returning to identity 1.1 and assumptions $(H_{1a,2c})$ we easily obtain

$$P_{f_i}(x_n, x_{n+1}) \le max\left\{d_n, d_{n+1}\right\},$$

and by assumptions $(H_{1a,1b,2a,2b})$, we get

$$\begin{aligned} P_{f_i}(x_n, x_{n+1}) &= f_i\left(d_n, d_n, d_{n+1}, \frac{1}{2}d(x_n, x_{n+2})\right) \\ &\ge max\left\{f_i\left(d_n, d_n, 0, \frac{|d_n - d_{n+1}|}{2}\right), f_i\left(0, 0, d_{n+1}, \frac{|d_n - d_{n+1}|}{2}\right)\right\}, \\ &\ge \delta_2\left(min\{d_n, d_{n+1}, \xi_1\} \cap \mathbb{R}^*\right), \end{aligned}$$

where $\xi_1 := liminf_{n\to\infty} d_n$. Consequently

$$\delta_2\left(min\left\{d_n, d_{n+1}, \xi_1^{\xi}\right\} \cap \mathbb{R}^*\right) \le P_{f_i}(x_n, x_{n+1}) \le max\left\{d_n, d_{n+1}\right\}, \quad (2.2)$$

for all $n \in \mathbb{N}$ and $i = 1, 2$. Doing the recap of the foregoing, we can assert that there exists a sub-sequence $\left\{d_{n_k}\right\}_k \subset \left\{d_n\right\}_n$ and $M, c \ge 0$ such that

$$d_{n_k} \le d_{n_k+1}, \quad d_k \le M, \quad \lim_{k\to\infty} d_{n_k} = \left(\le \liminf_{k\to\infty} d_{n_k+1}\right),$$

and

$$\delta_2 \frac{c}{2} \le \delta_2 min\left\{d_{n_k}, c\right\} \le P_{f_i}(x_{n_k}, x_{n_k+1}) \le d_{n_k+1} \le M,$$

for sufficiently large values of k and i = 1,2.. Using this last result and having in mind relations (1.2), (2.1) and condition A_1, we can write

$$\int_0^{d_{n_k+1}} \varphi(\mathrm{t})dt \le \theta\left(\int_0^{d_{n_k+1}} \varphi(\mathrm{t})dt\right) - \int_0^{\psi\left(P_{f_2}\left(x_{n_k},x_{n_k+1}\right)\right)} \varphi(t)dt.$$

Remember that $\delta_2 \frac{c}{2} \le P_{f_i}(x_{n_k}, x_{n_k+1}) \le M$, and in view the assumed lower semi-continuity of ψ, we deduce that there exists $a_1 \in [\delta_2 \frac{c}{2}, M]$ such that $\psi(a_1) = \inf_{x\in\left[\delta_2\frac{c}{2},M\right]} \psi(x)$, therefore

$$\int_0^{d_{n_k+1}} \varphi(t)dt \le \theta\left(\int_0^{d_{n_k+1}} \varphi(t)dt\right) - \int_0^{\psi(a_1)} \varphi(t)dt,$$

which is a contradiction with lemma (1.15). Thus the sequence $\{d_n\}_n$ is decreasing and it's bounded below by 0, hence $\{d_n\}_n$ converges to some $c \ge 0$, we write for sufficiently large values of n

$$d_n > d_{n+1}, \quad and \quad \lim_{n\to\infty} d_n = c. \tag{2.3}$$

Taking into account (2.3) and the double inequality (2.2), we obtain for sufficiently large values of n that $\delta_2 c \le P_{f_i}\left(x_n, x_{n+1}\right) \le 2c$, then there exists $a_2 \in [\delta_2 \mathrm{c}, 2\mathrm{c}]$ such that $\psi\left(a_2\right) = \inf_{x\in\left[\delta_2 c, 2c\right]} \psi\left(x\right)$. Now, supposing that $c > 0$. By virtue of (1.2) and (2.2) it follows that

$$\int_0^{d_{n+1}} \varphi\left(t\right)dt \le \Theta\left(\int_0^{d_n} \varphi\left(t\right)dt\right) - \int_0^{\psi(a_2)} \varphi\left(t\right)dt,$$

from which with relations (2.1), (2.3) and lemma (1.15) we deduce that $\psi(a_2) = 0$ and consequently c = 0, which means

$$\lim_{n\to\infty} d_n = 0. \tag{2.4}$$

Now, we need to assert that the sequence $\{x_n\}_n$ has the Cauchy property. Assume the contrary. Then, by virtue of the last limit, we can extract two subsequences $\left\{x_{n_k}\right\}_k$ and $\left\{x_{m_k}\right\}_k$ from the sequence $\{x_n\}_n$ such that

$$\exists \varepsilon > 0, \forall w_\varepsilon \le \frac{\varepsilon}{8}, \exists N_{w_\varepsilon} \in \mathbb{N}, \forall k \ge N_{w_\varepsilon}, \exists_{n_k} > m_k \ge k;$$

$$d\left(x_{n_k}, x_{m_k}\right) > \varepsilon, \tag{2.5}$$

with

$$d\left(x_{n_k-1}, x_{m_k}\right) \le \varepsilon, \tag{2.6}$$

and

$$d\left(x_i, x_{i+1}\right) < w_{\varepsilon N_{w_\varepsilon}}, \forall i \ge N_{w_\varepsilon}. \tag{2.7}$$

With consideration to estimates (2.5) and (2.7) we deduce that

$$\begin{aligned} d\left(x_{n_k-1}, x_{m_k-1}\right) &\ge d\left(x_{n_k}, x_{m_k}\right) - d\left(x_{n_k-1}, x_{m_k}\right) - d\left(x_{n_k}, x_{m_k-1}\right) \\ &\ge \varepsilon - \left[d\left(x_{n_k-1}, x_{m_k-1}\right) + d_{m_k-1}\right] - \left[d_{n_k-1} + d\left(x_{n_k-1}, x_{m_k-1}\right)\right] \\ &\ge \frac{\varepsilon - 2w_\varepsilon}{3} \ge \frac{\varepsilon}{4}, \end{aligned}$$

And, by (2.6), that

$$\begin{aligned} d\left(x_{n_k-1}, x_{m_k-1}\right) &\le d\left(x_{n_k-1}, x_{m_k}\right) + d_{m_k-1} \\ &\le \varepsilon + w_\varepsilon, \end{aligned}$$

Which give

$$\frac{\varepsilon}{4} \le d\left(x_{n_k-1}, x_{m_k-1}\right) \le \varepsilon + w_\varepsilon, \tag{2.8}$$

for all $k \ge N_{w_\varepsilon}$. Similarly we obtain

$$\begin{aligned} \frac{1}{2}\left[d\left(x_{n_k-1}, x_{m_k}\right) + d\left(x_{n_k}, x_{m_k-1}\right)\right] &\le d\left(x_{n_k-1}, x_{m_k-1}\right) + \frac{d_{n_k-1} + d_{n_k-1}}{2} \\ &\le \varepsilon + 2w_\varepsilon. \end{aligned}$$

Remark 2.2. This inequality implies that

$$\begin{aligned} \frac{1}{2}\liminf_{k\to\infty}\left[d\left(x_{n_k-1}, x_{m_k}\right) + d\left(x_{n_k}, x_{m_k-1}\right)\right] &\le \liminf_{k\to\infty} d\left(x_{n_k-1}, x_{m_k-1}\right) \\ &\le \varepsilon + 2w_\varepsilon. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{1}{2}\left[d\left(x_{n_k-1}, x_{m_k}\right) + d\left(x_{n_k}, x_{m_k-1}\right)\right] &\ge d\left(x_{n_k-1}, x_{m_k-1}\right) - \frac{d_{n_k-1} + d_{m_k-1}}{2} \\ &\ge \frac{\varepsilon}{8}. \end{aligned}$$

Then

$$\frac{\varepsilon}{8} \le \frac{1}{2}\left[d\left(x_{n_k-1}, x_{m_k}\right) + d\left(x_{n_k}, x_{m_k-1}\right)\right] \le \varepsilon + 2w_\varepsilon. \tag{2.9}$$

Now, thanks to relations (1.1), (2.4), (2.8), (2.9), remark 2.2, assumptions $(H_{1a,2c})$ and studying the two cases: $x \in \Pi_{f_i(x,y,z,t)}$ and $x \notin \Pi_{f_i(x,y,z,t)}$, it follows that

$$\begin{aligned} P_{f_i}\left(x_{n_k-1}\right)x_{m_k-1} &\le f_i\left(d\left(x_{n_k-1}, x_{m_k-1}\right), d_{n_k-1}, d_{m_k-1}, \frac{d\left(x_{n_k}, x_{m_k-1}\right) + d\left(x_{n_k-1}, x_{m_k}\right)}{2}\right) \\ &\le max\left\{d\left(x_{n_k-1}, x_{m_k-1}\right), \frac{d\left(x_{n_k}, x_{m_k-1}\right) + d\left(x_{n_k-1}, x_{m_k}\right)}{2}, \right. \\ &\qquad \left. f_i\left(d_{n_k-1}, d_{n_k-1}, d_{m_k-1}, \frac{d\left(x_{n_k}, x_{m_k-1}\right) + d\left(x_{n_k-1}, x_{m_k}\right)}{2}\right)\right\} \\ &\le \varepsilon + 2w_\varepsilon, \end{aligned} \tag{2.10}$$

for all $k \geq N_{w_\varepsilon}$. Let's suppose that $\{x,t\} \bigcap \prod_{f_i(x,y,z,t)} = \varnothing$, hence going back to relations (1.2), (2.1), (2.3), condition A_1 we obtain for $n < m$

$$\int_0^{d(x_{n+1},x_{m+1})} \varphi(t)dt \leq \Theta\left(\int_0^{P_{f_1}(x_n,x_m)} \varphi(t)dt\right) - \int_0^{\psi\left(P_{f_2}(x_n,x_m)\right)} \varphi(t)dt$$
$$\leq \Theta\left(\int_0^{f_1(d_n,d_n,d_n,d_n)} \varphi(t)dt\right) - \int_0^{\psi\left(P_{f_2}(x_n,x_m)\right)} \varphi(t)dt,$$

invoking assumption (H_{2c}) we derive

$$\int_0^{d(x_{n+1},x_{m+1})} \varphi(t)dt \leq \Theta\left(\int_0^{d_n} \varphi(t)dt\right).$$

Which –in view of relation (2.4) and remark 1.9 - means that the sequence $\{x_n\}_n$ has the Cauchy property. Assume now that $\{x,t\} \cap \prod_{f(x,y,z,t)} \neq \varnothing$. By (2.4), (2.8), (2.9), remark 2.2 and assumption (H_{2b}), we estimate

$$P_{f_i}(x_{n_k-1}, x_{m_k-1}) \geq f_i\left(d(x_{n_k-1}, x_{m_k-1}), d_{n_k-1}, d_{m_k-1}, \frac{d(x_{n_k}, x_{m_k-1}) + d(x_{n_k-1}, x_{m_k})}{2}\right)$$
$$\geq \delta_3 \min\left\{d(x_{n_k-1}, x_{m_k-1}), \frac{d(x_{n_k}, x_{m_k-1}) + d(x_{n_k-1}, x_{m_k})}{2}, \liminf_{k\to\infty} \frac{d(x_{n_k}, x_{m_k-1}) + d(x_{n_k-1}, x_{m_k})}{2}\right\}$$
$$\geq \delta_3 \frac{\varepsilon}{8}, \forall k \geq N_{w_\varepsilon} \tag{2.11}$$

From estimates (2.10)–(2.11) andand invoking the fact that ψ is a lower semi-continuous function, we deduce that there exists $a_3 \in \left[\delta_3 \frac{\varepsilon}{8}, \varepsilon + 2w_\varepsilon\right]$ such that, $\psi(a_3) = \inf_{x \in \left[\delta_3 \frac{\varepsilon}{8}, \varepsilon+2w_\varepsilon\right]} \psi(x)$ this together with relations (1.2), (2.5), (2.10) and condition A_1 yields

$$\int_0^{\varepsilon} \varphi(t)dt \leq \Theta\left(\int_0^{\varepsilon+2w_\varepsilon} \varphi(t)dt\right) - \int_0^{\psi(a_3)} \varphi(t)dt$$
$$\leq \Theta\left(\int_0^{\varepsilon} \varphi(t)dt + \int_{\varepsilon}^{\varepsilon+2w_\varepsilon} \varphi(t)dt\right) - \int_0^{\psi(a_3)} \varphi(t)dt$$
$$\leq \Theta\left(\left(\int_0^{\varepsilon} \varphi(t)dt\right)\left[1 + \left(\int_0^{\varepsilon+2w_\varepsilon} \varphi(t)dt \setminus \left(\int_0^{\varepsilon} \varphi(t)dt\right)\right)\right]\right) - \int_0^{\psi(a_3)} \varphi(t)dt$$

Which with condition A_2 gives

$$\int_0^{\varepsilon} \varphi(t)dt \leq \vartheta_2^r \Theta\left(\int_0^{\varepsilon} \varphi(t)dt\right) - \int_0^{\psi(a_3)} \varphi(t)dt,$$

where

$$\vartheta_2 := 1 + \left(\int_{\varepsilon}^{\varepsilon+2w_\varepsilon} \varphi(t)dt \Big/ \left(\int_0^{\varepsilon} \varphi(t)dt\right)\right) \to_{w_\varepsilon \to 0} 1^+.$$

Continuing in this way, we obtain

$$\int_0^{\varepsilon} \varphi(t)dt \leq \vartheta_2^{\sum_{i=1}^m r^i} \Theta^m\left(\int_0^{\varepsilon} \varphi(t)dt\right) - \int_0^{\psi(a_3)} \varphi(t)dt.$$

Having in mind conditions A_1 and A_2 we get for sufficiently small values of ε and w_ε

$$\int_0^{\varepsilon} \varphi(t)dt \leq \vartheta_2^{r\frac{1-r^m}{1-r}} \sup_{m\in\mathbb{N}}\left[\Theta^m(1)\right]\left(\int_0^{\varepsilon} \varphi(t)dt\right)^{mr} - \int_0^{\psi(a_3)} \varphi(t)dt$$
$$\leq \left(1 + \frac{1-r}{r(1-r^m)}\right)^{r\frac{1-r^m}{1-r}} \sup_{m\in\mathbb{N}}\left[\Theta^m(1)\right]\left(\int_0^{\varepsilon} \varphi(t)dt\right)^{mr} - \int_0^{\psi(a_3)} \varphi(t)dt$$

The right hand side of the previous inequality tends to $-\int_0^{\psi(a_3)} \varphi(t)dt$ while m $\to \infty$, or

$$\int_0^{\varepsilon} \varphi(t)dt \leq -\int_0^{\psi(a_3)} \varphi(t)dt,$$

which is a contradiction with the fact that $\varepsilon > 0$ *and* $(\varphi, \psi) \in \Phi_1 \times \Phi_2$. This contradiction, with the fact that X is complete - leads to the result ($\{x_n\}_n$ is a convergent sequence). Consider

$$a := \lim_{n\to\infty} x_n. \tag{2.12}$$

Remark 2.3. With this last result we can assert that there exists a subsequence $\{x_{m_k}\}_k$ from $\{x_n\}_n$ satisfying the following property:

$$d(x_{m_k}, a) \geq \frac{1}{2} d(x_{m_k}, x_{m_k+1}) \tag{2.13}$$

for all $k \in \mathbb{N}$.

Let us show that a is a fixed point of T. Assume the contrary. Using relations (1.1), (2.4), (2.12), (2.13) and assumptions $(H_{1b,2c})$, we obtain for sufficiently large values of k

$$P_{f_i}(x_{m_k}, a) = f_i\left(d(x_{m_k}, a), d_{m_k}, d(a,Ta), \frac{d(x_{m_k}, Ta) + d(x_{m_k+1}, a)}{2}\right)$$
$$\leq d(a,Ta) \tag{2.14}$$

On the other hand, thanks to limits (2.4), (2.12) and assumptions $(H_{1b,2c})$ we obtain

$$P_{f_i}(x_{m_k}, a) = f_i\left(d(x_{m_k}, a), d_{m_k}, d(a,Ta), \frac{d(x_{m_k}, Ta) + d(x_{m_k+1}, a)}{2}\right) \tag{2.15}$$
$$\geq \delta_4 d(a,Ta)$$

The two estimates (2.14) and (2.15) mean that $P_{f_i}(x_{m_k}, a) \in \left[\delta_4 d(a,Ta), d(a,Ta)\right]$, then there exists $a_4 \in \left[\delta_4 d(a,Ta), d(a,Ta)\right]$ such that

$$\psi(a_4) = \inf_{x \in [\delta_4 d(a,Ta), d(a,Ta)]} \psi(x).$$

This result with (1.2), (2.14) and condition A_1 imply that

$$a(x_{m_k}, a)\int_0^{d(x_{m_k+1},Ta)} \varphi(t)dt \le \Theta\left(\int_0^{d(a,Ta)} \varphi(t)dt\right) - \int_0^{\psi(a_4)} \varphi(t)dt,$$

for sufficiently large values of k. Observing that

$$d(x_{m+1}, Ta) \ge d(a, Ta) - d(x_{m_k+1}, a)$$

The previous inequality gives

$$a(x_{m_k}, a)\int_0^{d(a,Ta)} \varphi(t)dt \le \Theta\left(\int_0^{d(a,Ta)} \varphi(t)dt\right) + a(x_{m_k}, a)\int_{d(a,Ta)-d(x_{m_k+1},a)}^{d(a,Ta)} \varphi(t)dt - \int_0^{\psi(a_4)} \varphi(t)dt,$$

taking into account thelimit (2.12) and lemma 1.14, this last inequality become

$$a(x_{m_k}, a)\int_0^{d(a,Ta)} \varphi(t)dt \le \Theta\left(\int_0^{d(a,Ta)} \varphi(t)dt\right) - \frac{1}{2}\int_0^{\psi(a_4)} \varphi(t)dt.$$

for sufficiently large values of k. Using again the fact that $lim_{k\to\infty} x_{m_k} = a$ and that $a \in \Gamma(T)$, we obtain

$$\int_0^{d(a,Ta)} \varphi(t)dt \le \Theta\left(\int_0^{d(a,Ta)} \varphi(t)dt\right) - \frac{1}{4}\int_0^{\psi(a_4)} \varphi(t)dt.$$

This is a contradiction with lemma 1.15. Then a is a fixed point of T. To conclude the proof, we need to show that under assumption: "for all fixed points x and y of the mapping T, we have $\alpha(x, y) \ge 1$", a is the unique fixed point of T. Assume the contrary, i.e.

$$\exists b \ne a \in X; Ta = a, Tb = b \text{ and } \alpha(a,b) \ge 1.$$

Going back to 1.2 we can write

$$a(x_n, b)\int_0^{d(x_{n+1},b)} \varphi(t)dt = \alpha(x_n, b)\int_0^{d(Tx_n,Tb)} \varphi(t)dt \le \Theta\left(\int_0^{P_{f_1}(x_n,b)} \varphi(t)dt\right) - \int_0^{\psi(P_{f_2}(x_n,b))} \varphi(t)dt. \quad (2.16)$$

Noting first that if $\{x,t\} \cap \Pi_{f(x,y,z,t)} = \varnothing$, by (1.2) and $(H_{1a,2c})$, relation (2.16) become

$$a(x_n, b)\int_0^{d(x_{n+1},b)} \varphi(t)dt \le \Theta\left(\int_0^{f_1(d_n,d_n,0,d_n)} \varphi(t)dt\right) - \int_0^{\psi(P_{f_2}(x_n,b))} \varphi(t)dt \le \Theta\left(\int_0^{f_1(d_n,d_n,d_n,d_n)} \varphi(t)dt\right) - \int_0^{\psi(P_{f_2}(x_n,b))} \varphi(t)dt \le \Theta\left(\int_0^{d_n} \varphi(t)dt\right) - \int_0^{\psi(P_{f_2}(x_n,b))} \varphi(t)dt$$

Having in mind that $\liminf_{n\to\infty} \alpha(x_n, b) \ge 1$ we get

$$\exists N \in \mathbb{N}, \forall n \ge N; \alpha(x_n, a) \ge 1$$

From which, with the fact that $\alpha(a, b) \ge 1$ we deduce that

$$\alpha(x_n, b) \ge 1 \quad (2.17)$$

for sufficiently large values of n. Using this result, lemma 1.12, remark 1.9 and passing to the limit as n tends to infinity in the last inequality, we get

$$\int_0^{d(a,b)} \varphi(t)dt \le 0,$$

which is a contradiction with the fact that $a \ne b$. Then we can suppose that

$$\{x,t\} \cap \Pi_{f(x,y,z,t)} \ne \varnothing. \quad (2.18)$$

Relations (1.1), (2.4) together with assumptions $(H_{1a,2c})$ allow us to write

$$P_{f_i}(x_n, b) = f_i\left(d(x_n,b), d_n, 0, \frac{1}{2}\left[d(x_n,b) + d(x_{n+1},b)\right]\right) \le max\left\{d(x_n,b), \frac{1}{2}\left[d(x_n,b) + d(x_{n+1},b)\right], f_i\left(d_n, d_n, 0, \frac{1}{2}\left[d(x_n,b) + d(x_{n+1},b)\right]\right)\right\} \le max\left\{d(x_n,b), \frac{1}{2}\left[d(x_n,b) + d(x_{n+1},b)\right], f_i(\xi,\xi,\xi,\xi)\right\}$$

Where

$$\xi = \max\left\{d(x_n,b), d(x_{n+1},b)\right\}.$$

Therefore

$$P_{f_i}(x_n, b) \le \max\left\{d(x_n,b), d(x_{n+1},b)\right\} \le 2d(a,b), \quad (2.19)$$

for sufficiently large values of n and $i = 1, 2$, which means that the sequence $\left\{P_{f_2}(x_n,b)\right\}_n$ is bounded. On the other hand, thanks to relations (1.1)(2.19) and assumption (H_{2b}) we deduce that

$$P_{f_2}(x_n, b) = f_2\left(d(x_n,b), d_n, 0, \frac{1}{2}\left[d(x_n,b) + d(x_{n+1},b)\right]\right) \ge \delta_5 \min\left\{d(a,b), d(x_n,b), \frac{1}{2}\left[d(x_n,b) + d(x_{n+1},b)\right]\right\}$$

from which

$$P_{f_2}(x_n,b) \geq \frac{\delta_5}{2}d(a,b).$$

Then, there exists $a_5 \in \left[\frac{\delta_5}{2}d(a,b),2d(a,b)\right]$ such that $\psi(a_5)=\inf_{x\in\left[\frac{\delta_5}{2}d(a,b),2d(a,b)\right]}\psi(x)$. Now, recalling relations (2.15), (2.16), (2.18) and examining the two cases $d(x_{n+1},b)\geq d(x_n,b)$ and $d(x_n,b)\geq d(x_{n+1},b)$, we get either

$$\int_0^{d(x_{n+1},b)}\varphi(t)dt \leq \Theta\left(\int_0^{d(x_{n+1},b)}\varphi(t)dt\right)-\int_0^{\psi(a_5)}\varphi(t)dt$$

Or

$$\int_0^{d(x_{n+1},b)}\varphi(t)dt \leq \Theta\left(\int_0^{d(x_n,b)}\varphi(t)dt\right)-\int_0^{\psi(a_5)}\varphi(t)dt,$$

for sufficiently large values of n. From where, we get a contradiction with Lemma 1.3. This finishes the proof.

3. Application

From the main theorem, we can easily get the following corollaries (we omit its proof for simplicity):

Corollary 3.1. Let (φ, ψ) be in $\Phi_1\times\Phi_2$ and T be a mapping from a complete metric space (X, d) into itself satisfying the integral inequality:

$$\int_0^{d(Tx,Ty)}\varphi(t)dt \leq \int_0^{N(x,y)}\varphi(t)dt-\int_0^{\psi(M(x,y))}\varphi(t)dt,$$

for all x, y in X. Where

$$N(x,y)=a_1d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}+a_2\frac{d(x,Tx)d(y,Ty)}{d(x,y)}+a_3d(x,y),$$

if $x \neq y$ and $N(x,y)=d(x,y)+d(y,Ty)$ if $x=y$, $(a_1,a_2,a_3)\in\mathbb{R}_+\times\left(\mathbb{R}_+^*\right)^2$ are constants with $a_1+2a_2+a_3<1$ and

$$M \in \{M_1, M_2. M_3, M_4, M_5, M_6\}.$$

Then T has a unique fixed point $a \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}_n$ converges to a.

Remark 3.2. This corollary extend both main results in[3,9] at the case where mapping T is not continuous.

Corollary 3.3. Let (φ, ψ) be in $\Phi_1\times\Phi_2$ and T be a mapping from a complete metric space (X, d) into itself satisfying the integral inequality

$$\int_0^{d(Tx,Ty)}\varphi(t)dt \leq \int_0^{N(x,y)}\varphi(t)dt-\int_0^{\psi(M(x,y))}\varphi(t)dt,$$

for all x, y in X. Where

$$N(x,y)=\max\left\{d(x,y),\frac{d(x,Tx)d(y,Ty)}{d(x,y)},d(y,Ty)\right\},$$

if $x \neq y$ and $N(x, y) = d(x, y) + d(y, Ty)$ if $x = y$ and

$$M \in \{M_1, M_2. M_3, M_4, M_5, M_6\}.$$

Then T has a unique fixed point $a \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}_n$ converges to a.

Corollary 3.4. L et (φ, ψ) be in $\Phi_1 \times \Phi_2$ and T be a mapping from a complete metric space (X, d) into itself satisfying the integral inequality

$$\int_0^{d(Tx,Ty)}\varphi(t)dt \leq \int_0^{N(x,y)}\varphi(t)dt-\int_0^{\psi(M(x,y))}\varphi(t)dt,$$

for all x, y in X. Where

$$N(x,y)=a_1d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}+a_2\sqrt{d(x,y)d(x,Tx)\frac{d(x,Tx)+d(y,Ty)}{d(x,y)+d(y,Ty)}}+a_3d(x,y),$$

if $x \neq y$ and $N(x, y) = d(x, y) + d(y, Ty)$ if $x = y$, $(a_1,a_2,a_3)\in\mathbb{R}_+\times\left(\mathbb{R}_+^*\right)^2$ are constants with $a_1 + a_2 + a_3 < 1$ and

$$M \in \{M_1, M_2. M_3, M_4, M_5, M_6\}.$$

Then T has a unique fixed point $a \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}_n$ converges to a.

Corollary 3.5. Let (φ, ψ) be in $\Phi_1\times\Phi_2$ and T be a mapping from a complete metric space (X, d) into itself satisfying the integral inequality

$$\int_0^{d(Tx,Ty)}\varphi(t)dt \leq \int_0^{N(x,y)}\varphi(t)dt-\int_0^{\psi(M(x,y))}\varphi(t)dt,$$

for all x, y in X. Where

$$N(x,y)=2a_1d(x,Tx)\frac{|d(x,Tx)-d(x,y)|}{d(x,Ty)+d(Tx,y)}+a_2d(y,Ty)+a_3d(x,y),$$

if $x \neq y$ and $N(x, y) = d(x, y) + d(y, Ty)$ if $x = y$, $(a_1,a_2,a_3)\in\mathbb{R}_+\times\left(\mathbb{R}_+^*\right)^2$ are constants with $a_1 + a_2 + a_3 < 1$ and

$$M \in \{M_1, M_2. M_3, M_4, M_5, M_6\}.$$

Then T has a unique fixed point $a \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}_n$ converges to a.

4. References

1. Branciari A. A fixed point theorem for mappings satisfying a general contractive condition of integral type. Int. J. Math. Math. 2002; 29(9):531–36.
2. Chatterjee H. On generalization of Banach contraction principle. Indian J. Pure Appl. Math. 1979; 10(4):400–403.
3. Dass BK, Gupta S. An extension of Banach contraction principle through rational expression. Indian J. Pure Appl. Math. 1975; 6(12):1455–58.
4. Gupta AN, Saxena A. A unique fixed point theorem in metric spaces. Math. Stud. 1984; 1984(52):156–58.
5. Jachymski J. Remarks on contractive conditions of integral type. Nonlinear Anal. 2009; 71(3-4):1073–81.
6. Jaggi DS. Some unique fixed point theorems. Indian Journal of Pure and Applied Mathematics. 1977; 8(2):223–30.
7. Liu Z et al. Some fixed point theorems for mappings satisfying contractive conditions of integral type. Fixed Point Theory Appl. 2014; 2014(69):14 pages.
8. Liu Z et al. Fixed point theorems for mappings satisfying contractive conditions of integral type and applications. Fixed Point Theory Appl. 2011; 2011(64):18 pages.
9. Pathak HK. Integral Φ-type contractions and existence of continuous solutions for nonlinear integral inclusions. Nonlinear Anal. 2009; 71(7-8):2577–91.
10. Rhoades BE. Two fixed-point theorems for mappings satisfying a general contractive condition of integral type. Int. J. Math. Math. Math. Sci. 2003; 2003(63):4007–13.
11. Rhoades BE. Some theorems on weakly contractive maps. Nonlinear Anal. 2001; 47(4):2683–93.
12. Suzuki T. Meir-Keeler contractions of integral type are still Meir-Keeler contractions. Int J Math Math. 2007; 2007(Article ID 39281): 6 pages.
13. Vishal G, Ashima K. Fixed Point Theorem in Fuzzy Metric Spaces Satisfying E. A Property, Indian Journal of Science and Technology. 2012; 5(12):3767–69.
14. Vetro C. On Branciari's theorem for weakly compatible mappings. Appl. Math. Lett. 2010; 23(6):700–05.