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Improving Information Content in Compressed Sensing by Modifying the Random Re-Construction Matrices

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Abstract

Background/Objectives: Compressed Sensing (CS) is an efficient sensing paradigm which guarantees reasonable reconstruction with less number of samples. We aim to increase the reconstruction quality of signals in CS. **Methods/Statistical Analysis: T**he behavior of random matrices is analyzed and an efficient method for improving the reconstruction quality is developed in CS based ECG reconstruction applications. The method is compared against Biorthogonal wavelet based approaches. **Findings:** Our analysis reveals that introduction of a modified column vector in the reconstruction matrix, which contains the sum of all columns of random matrix increases the reconstruction quality in CS applications. This idea was applied to different sparsifying domains and the results are very encouraging. We studied the effect of doing this on the singular values and both unitary matrices U and V. The first singular value (**Σ**) shot up making the condition number high, however there was not much change in the other singular values. The matrix U seems to remain random unitary matrix, where as matrix V has one value becoming unity in its rank space. **Application/Improvements**: Compared to wavelet based approaches the method shows reasonable improvement in Percentage Root Square Deviation (PRD).

Keywords: Compressed Sensing, ECG, PRD, Singular Values, Splines

1. Introduction

It was emphasized that a paradigm shift is necessary to explore the intricacies of Compressed Sensing (CS)[1]. It has also been shown[2] that how the random matrix can be modified by use of partial diagonal matrices in the reconstruction space. In the present paper, we have found that modifying a column without changing the structure of the matrix improves the results further. We were able to get more than 50% of improvement in reconstruction quality. To best of our knowledge this is best result obtained for signal like ECG (raw data without filtering or thresholding). Filtering of the ECG signal and thresholding during reconstruction was not attempted, since a realistic comparison for the analytical methods were found necessary.

The paper is arranged as follows. Section 2 describes brief review of Compressed Sensing, Section 3 explains how to create sparsifying matrices and in subsection we describe how we created sparsifying matrices from exponential splines. In this section we elaborate results which illustrate exponential spline basis are better than Biorthogonal wavelet basis for ECG reconstruction in compressed sensing applications. In Section 4 we extend the idea of duplication on two types of reconstruction matrices, one made using E-Spline bases and the other using Biorthogonal basis. This section details the main contribution of this paper which details how to duplicate

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a column with the sum of others columns. Results are discussed in Section 5. We conclude this paper in Section 6.

2. Compressed Sensing

CS[3–5] allows us to reconstruct the original data from less number of random projections. Most of the information that are captured are thrown away in normal compression techniques, in a sense the effort to capture the information is wasted, but in CS only the required amount of information is captured. In CS, input data (N x 1) is projected on a random matrix (m x N) m<<N and the projected samples (m x 1) are used for reconstruction. If the data is not sparse in the domain of sensing, a sparsifying basis matrix of size N x N is used. The random sensing matrix is multiplied with the sparsifying matrix. We had done our modification on the resultant matrix. This matrix of size m x N is often known as reconstruction matrix. The complete CS scenario can be explained simply as, If ‘ is the input vector of size N x 1 and if, it has alternative notation in ψ domain where it is sparse that is $x = \psi\alpha$, then CS theory states that ‘*x*’ can be recovered from the vector ‘*y*’ of size m x 1 (m<<N) by solving convex optimization problem of type min $||\alpha||_{11}$s.t.y = $\Phi\alpha\cdot||\alpha||_{11}$. where ’*y*’ is the projection of ‘*x*’ on φ. i.e. $y = \varphi x = \varphi\psi\alpha = \Phi\alpha.\cdot\alpha$ is a column vector which are the coefficients of ‘*x*’ in ψ. We had done our modification in Φ matrix which is obtained by multiplying random sensing matrix φ with the sparsifying basis matrix Signals like ECG in its raw form (with noise) are not sparse in any domain (without filtering and thresholding). Since in compressed sensing techniques prior information of the signals are not supposed to be known, here also filtering and thresholding are not attempted.

3. Sparsifying Matrix (Ψ) Creation

Normally Biorthogonal wavelets are used as sparsifying basis in CS applications for better results[6] especially for ECG applications. We had developed a set of basis from Exponential Splines (E-Spline) and found out that E-Spline basis outperform Biorthogonal wavelet basis. Comparison results are given in the subsequent sections.

3.1 Sparsifying Matrix from Exponential Splines

Splines are polynomial segments connected in smooth faction. Exponential Splines are obtained by connecting one sided exponential. Michael Unser details about exponential splines[7], we have followed the same way and made 1st, 2nd and 3rd order splines. Higher order splines can be made from the successive convolution of lower order ones. The splines obtained are shown in Figure 1. The representative magnitudes selected from 1st, 2nd and 3rd order splines are:

F1 = [0 0.2009 0.4089 0.7704 1 0.7704 0.4089 0.2009 0]
F2 = [0 0.2445 0.5469 0.8647 1 0.8647 0.5469 0.2445 0]
F3 = [0 0.3427 0.6286 0.8913 1 0.8913 0.6286 0.3427 0]

For each of these we had made sparsifying matrices and labeled them as 1E1, 1E2, 1E3 and conducted CS based reconstruction using each of them on an ECG data obtained from MIT data base[8] of size 800 x 1. The ECG is shown in Figure 2. Results are compared against the

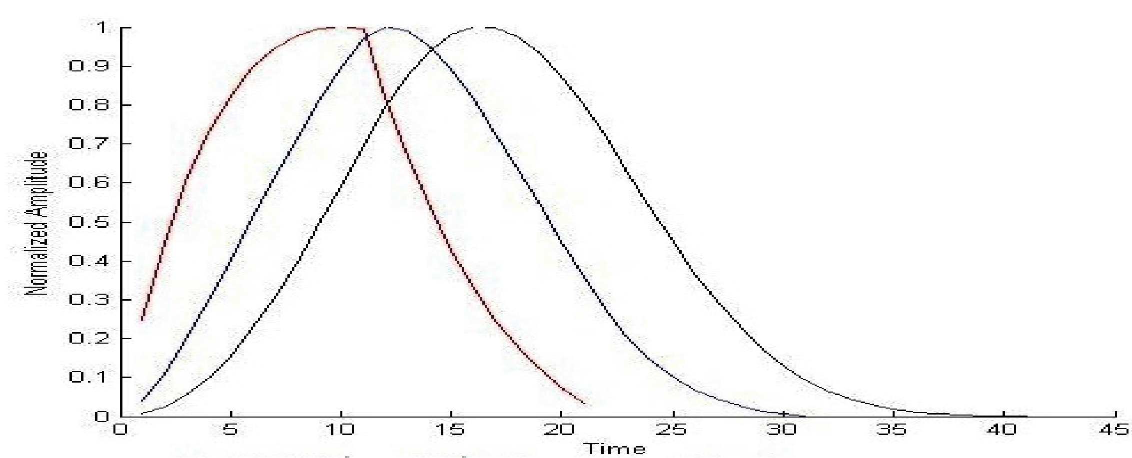

Figure 1. Exponential Splines of first three orders.

signal reconstructed using a Biorthogonal wavelet matrix and all these came better than that and it is shown in Figure 6. This is our primary level results which show E-Splines are better than Biorthogonal wavelets for CS based ECG reconstruction. In our second level we extend our idea of duplication in the reconstruction matrices made from both E-Spline and Biorthogonal wavelets. The corresponding F1, F2 and F3 are graphically shown in Figure 3, Figure 4 and Figure 5.

Reconstruction quality is analyzed on the basis of Percentage Root Mean Square Deviation (PRD).

$$\mathrm{PRD} = \frac{\sum_{n=1}^{N}\left(Xi(n) - Xr(n)\right)^2}{\sum_{n=1}^{N} Xi(n)^2} \times 100 \qquad (1)$$

Where $x_{(in)}$ the input is signal and $x_{(r)}$ is the signal after reconstruction.

1E1 indicate the reconstruction using the sparsifying matrix made from F1, 1E2 indicates the reconstruction using the matrix made F2 and 1E3 from the matrix from F3. All these are compared against when Biorthogonal wavelet is used as the sparsifying basis. All the PRDs came lesser than that of bior 4.4. Number of measurements (random projections) ranges from 10% of 800 that is from 80 to 75% (600). These results confirm us that E-Spline basis are better than Biorthogonal wavelet basis for compressed sensing for ECG signals, so we have introduced the idea of duplication in the reconstruction matrix made using E-Splines. In order to validate our results we have done the same type of duplication in reconstruction matrix made using Biorthogonal wavelet basis. There also the results improved compared to the unmodified case. These results are shown in Section 5.

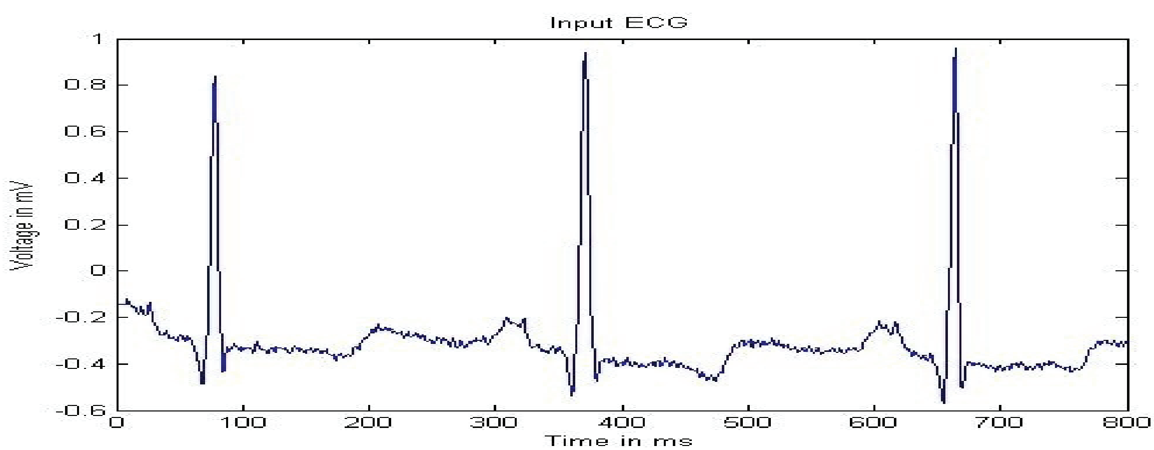

Figure 2. Input ECG.

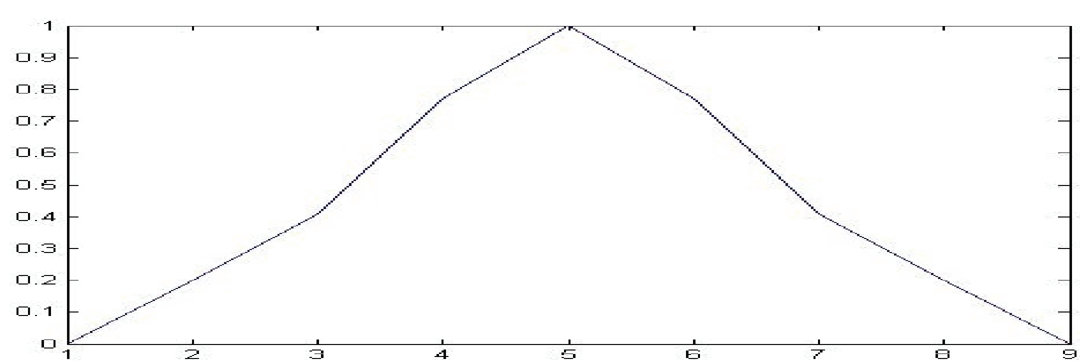

Figure 3. Representative cordinate F1.

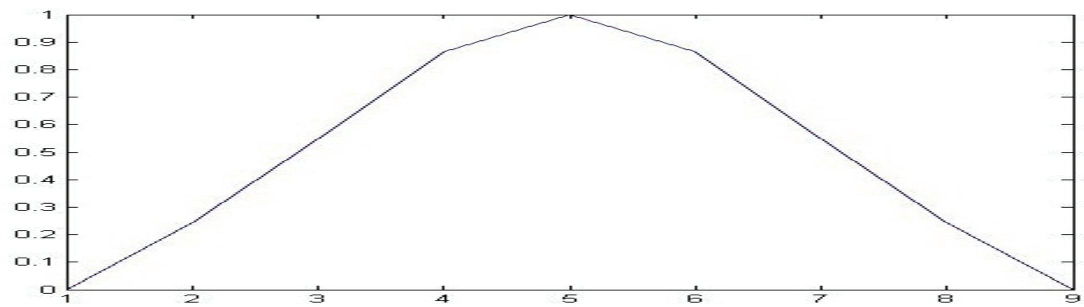

Figure 4. Representative cordinates F2.

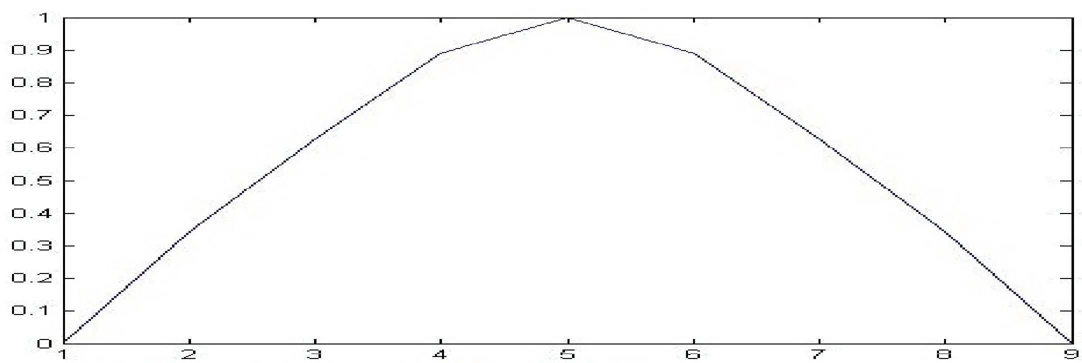

Figure 5. Representative Cordinates F3.

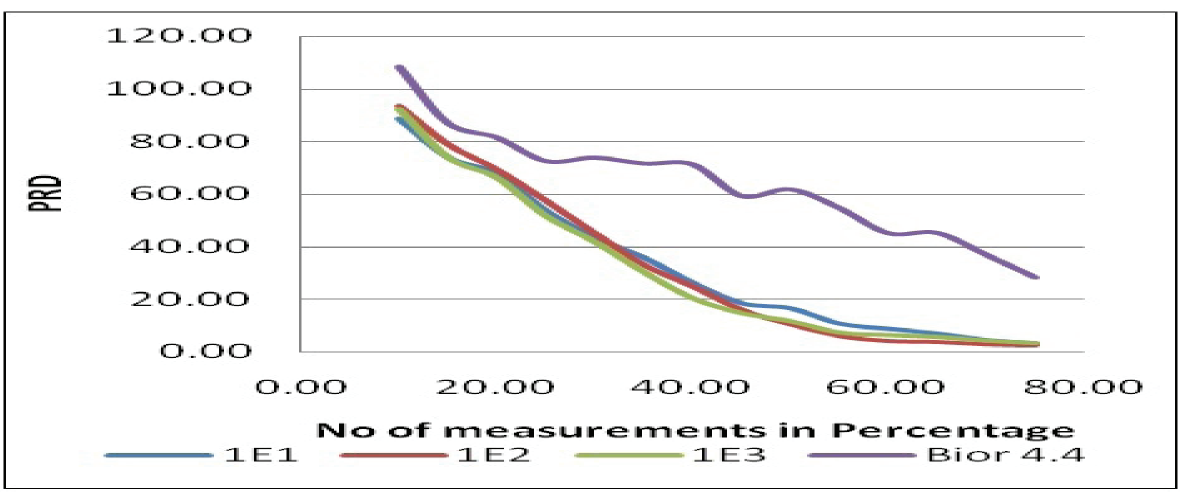

Figure 6. Comparison on exponential Splines and biorthogonal wavelet matrix reconstructions.

4. Proposed Method

Maximally incoherent column vectors in reconstruction matrix (Φ) guarantees a unique recovery in Compressed Sensing applications. The mutual coherence or simply coherence measures the maximum similarity between the vectors. Coherence for a matrix A with columns as a_1, $a_2 \ldots a_m$ is given as:

$$M = \max\left|a_i^H, a_j\right|.$$
$$\text{for } i < 1, j < m$$

It is basically taking the projection (dot product) between two vectors and varies in between 0 and 1. For increasing the coherence we had replaced one column vector in the reconstruction matrix by the sum of columns of the random matrix (φ). This can be done putting a column of ones in the sparsifying matrix. This is happening because, during matrix multiplication when a row is projected on to a column vector containing all ones, the resultant will be the sum of the projected vector. If a particular column of (say j^{th}) column of the sparsifying basis matrix is having all ones (it is not necessary to be one any other integer will give the result), j^{th} column of the reconstruction matrix will be the linear combination of all other columns of the random matrix, hence we give the name duplication.

Figure 7 and Figure 8 shows two matrices Φ_1 (400 X 800) and Φ_2 (400 X 800), where in Φ_2 all elements in 400th column are replaced by the sum of columns of the random sensing matrix (φ). The dot product between two consecutive vectors comes around .8 to .9. The dot product between two consecutive vectors in unmodified case comes around .8 to .9 where in modified case the dot product comes less than 0.1. Scaling certain vectors in reconstruction vectors will increase the incoherence but while reconstruction we must multiply the same sparsifying matrix to get back the signal as, x = , so our method of modification of column vector in sparsifying matrix by ones will help to reconstruct the signal exactly.

5. Result and Discussions

Table 1 compares the results with modified reconstruction matrix (Φ_2) with unmodified matrix for the input signal shown in Figure 2, which is of normal ECG type obtained from MIT data base[8] record 103m. We had tried with abnormal ECG signals also, which is shown in Figure 9. The results obtained for this signal in Figure 9 is shown in Table 2. Here also the improvement was substantial.

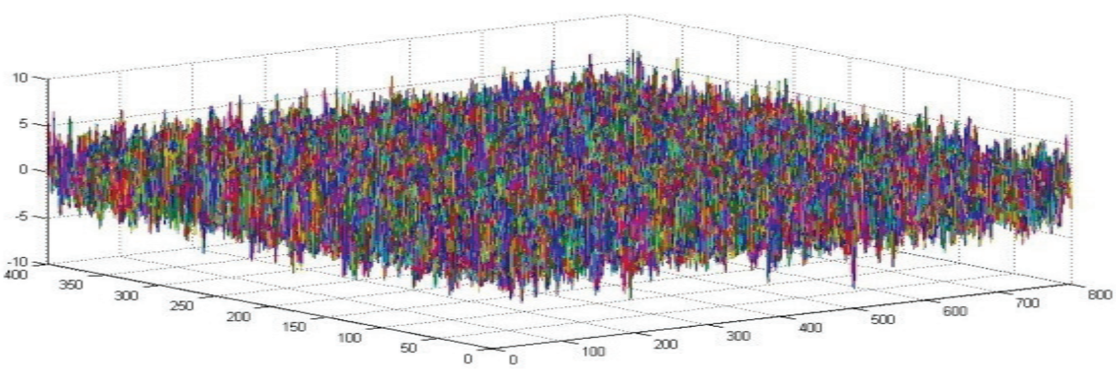

Figure 7. $\Phi_1 = \varphi\psi$, where φ is the random sensing matrix and ψ is the sparsifying matrix.

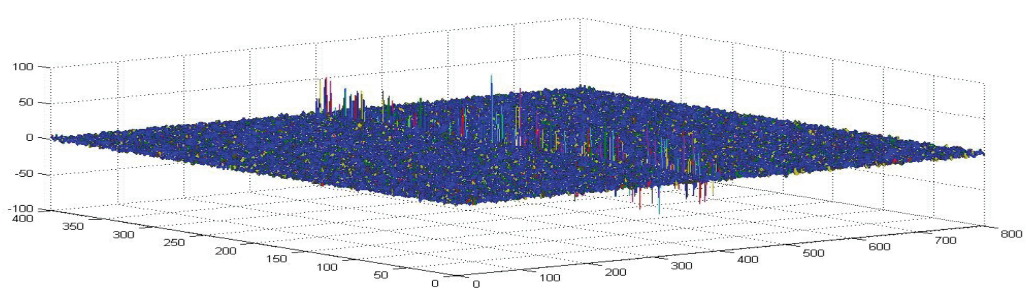

Figure 8. Φ_2, where column sum of φ replace the 400th column.

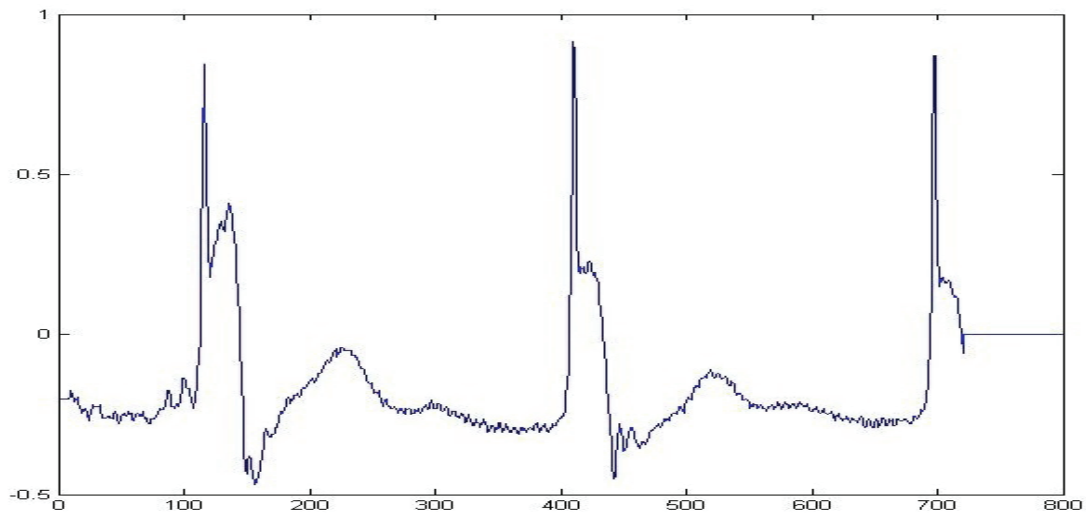

Figure 9. Abnormal ECG [9].

Table 1. Random samples vs. PRD

No: of Random Samples taken from 800 x1 input vector .	PRD-1E1	PRD-1E2	PRD-1E3	PRD-Bior 4.4	PRD after modifcation for 1E1	PRD after modifcation for 1E2	PRD after modifcation for 1E3	PRD after modifcation for Bior4.4
80.00	88.81	93.32	92.38	108.35	20.4994	21.2349	20.2137	50.8681
120.00	74.16	78.79	74.02	86.88	15.7661	16.5257	15.4717	38.5456
160.00	67.88	69.24	66.25	81.47	11.5477	11.0857	12.0057	28.425
200.00	53.81	57.50	51.79	72.52	9.7007	9.3771	9.1115	22.6069
240.00	43.51	45.07	41.86	73.84	6.5692	7.0968	6.3339	21.9363
280.00	35.93	33.09	30.34	71.56	5.731	5.7996	5.856	17.5538
320.00	26.32	24.90	20.39	71.22	4.4007	4.8104	4.4359	15.7269
360.00	18.56	16.08	14.72	59.23	3.868	4.1227	3.8131	15.5494
400.00	16.62	10.60	11.65	61.78	3.5319	3.5452	3.0002	11.3524
440.00	10.82	6.19	7.31	54.67	3.4017	3.267	2.6602	10.8858
480.00	8.83	4.26	6.37	45.11	3.0745	3.1139	2.5387	8.7638
520.00	7.03	3.90	5.70	45.26	3.0476	2.8674	2.527	7.3366
560.00	4.57	2.99	4.25	36.93	2.5604	2.7856	2.2834	6.5067
600.00	3.35	2.63	3.26	28.25	2.4176	2.2687	2.1417	4.7271

Clearly at an average of 50% of increase in reconstruction quality is there in modified method. In reconstructions side reconstruction algorithms basically returns the coefficients α which are the coefficients of ‘*x*’ in ψ. *x* can be reconstructed as $x = \psi\,\alpha$.

In Tables 1 and 2, 1E1 indicates when 1st order spline coordinates (F1) are used for making the sparsifying matrix. 1E2 for F2 and 1E3 for F3. Table indicates the difference in PRDs when the modification is done in measurement matrix while using these spline coordinates. Our experiments show that there is substantial improvement in PRD even for Biorthogonal wavelet basis when comparing modified and unmodified cases.

The graph in Figure 10 dictates the improvement in PRD before and after modification in measurement

Table 2. Random samples vs. PRD

No: of Random Samples taken from 800 x1 input vector .	PRD-1E1	PRD-1E2	PRD-1E3	PRD-Bior 4.4	PRD after modifcation for 1E1	PRD after modifcation for 1E2	PRD after modifcation for 1E3	PRD after modifcation for Bior4.4
80	86.41	89.67	87.62	107.11	54.6236	54.218	54.9216	77.1839
120	75.89	80.16	76.87	84.58	43.427	44.4456	41.9154	59.4779
160	64.31	69.94	63.53	73.10	27.6844	28.7917	27.6138	53.9111
200	50.39	53.78	49.86	67.46	21.6142	20.4679	21.8589	51.2171
240	38.13	40.49	32.92	60.27	17.5227	15.1197	14.8688	44.3655
280	27.17	27.10	23.36	58.95	14.5978	10.6096	12.0841	44.8997
320	21.52	19.79	16.19	54.34	12.7238	8.7932	11.6678	37.8369
360	16.18	11.49	12.29	51.33	10.2379	7.9668	8.9614	32.2924
400	11.02	8.09	9.40	40.23	8.8409	7.0405	7.4815	31.2757
440	11.14	6.49	8.77	40.83	8.7917	6.0505	6.9935	26.3513
480	9.12	5.90	7.91	30.51	7.3567	5.2896	6.8954	24.6267
520	8.08	5.92	6.63	28.87	7.908	5.9754	6.4532	20.4767
560	7.93	5.43	5.76	16.66	7.4984	5.3355	5.7352	16.4755
600	7.02	5.11	5.13	8.52	6.7851	5.1951	5.1825	13.4541

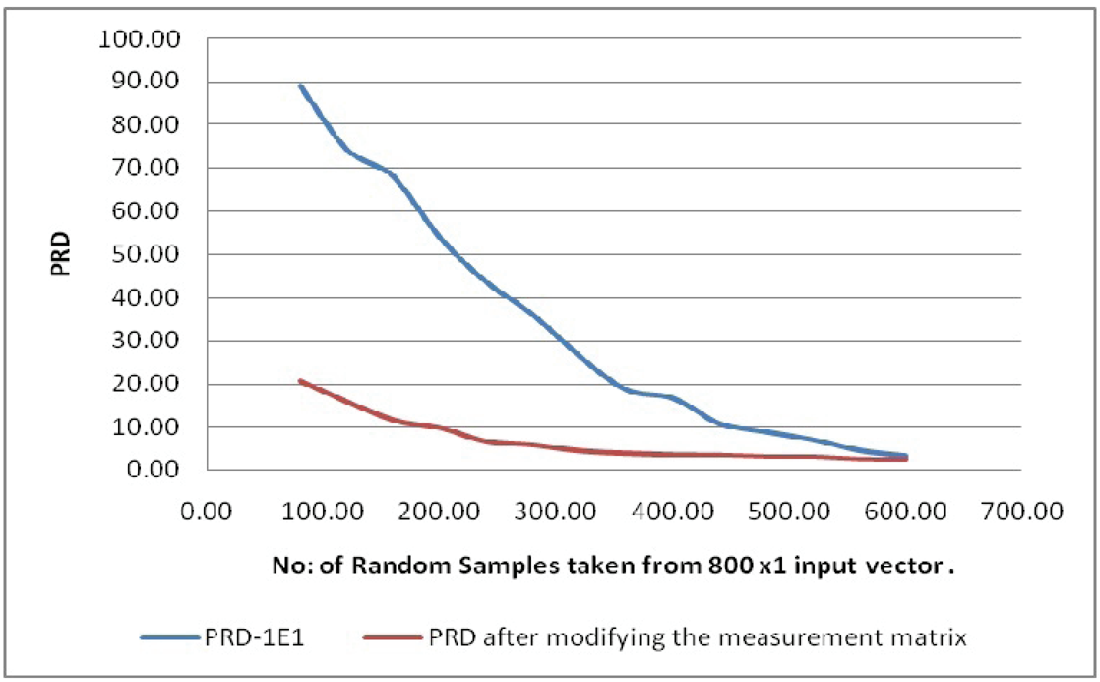

Figure 10. Comparison between proposed method and normal reconstruction using first order spline as transform (sparsifying) basis.

matrix when First order spline (1E1) coordinates are used for making the sparsifying matrix, for the input signal shown in Figure 2. The Figure 11 shows the improvement when Biorthogonal wavelets are used as the transform basis for the input signal in Figure 2 for modified and unmodified cases. For further insight we had done SVD and QR in modified and unmodified matrices.

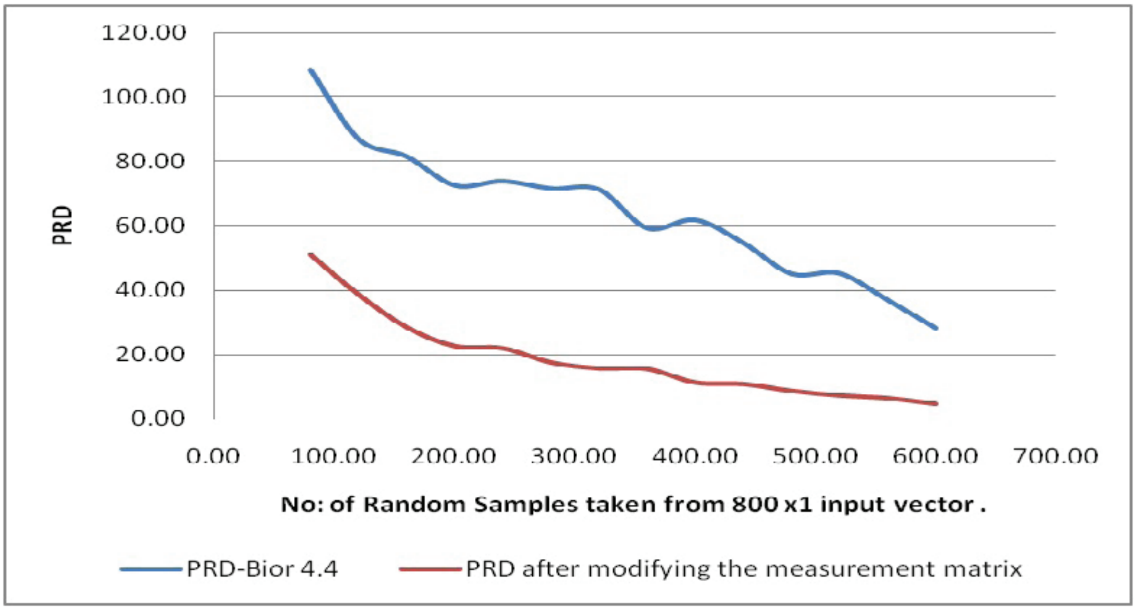

Figure 11. Comparison between proposed method and normal reconstruction using bi orthogonal wavelet as transform (sparsifying) basis.

5.1 Singular Value Decomposition – SVD

SVD for a real matrix A (m x N) , is given by $U\Sigma V^T$, where *U* is an 'm x m' matrix whose 'm' column are called left singular values, which are the Eigen vectors of AA^T. *V* is an N x N matrix whose first N columns are called as right singular values, which are the Eigen vectors of A^TA. **Σ** has the Eigen values from A^TA. They are called the singular values of A. If A has a rank r, then there will be r singular values, rest of the values will be zero. From U and V all fundamental subspaces can be found, first r columns of U are bases for the column space of A, remaining 'm-r' columns are the basis of left null space of A. (Null space of A^T). First 'r' columns in V are the bases for the row space of A where last 'N-r' columns in V give the bases for null space of A.

5.1.1 SVD of an Unmodified Reconstruction Matrix

For an unmodified Φ (Φ = φψ), with m rows and N columns and with all independent rows (rank, r = m), the 3 dimensional plot (2 dimensional view didn't reveal much details) of U and V are shown in Figure 12 and Figure 13. The matrix is of size 520 x 800. Figure 13 presents the V matrix of an unmodified matrix, as we had considered A matrix of size 520 x 800, first 520 (m) columns are the bases of row space remaining 280 (N-r, 800-520) columns are the bases of null space of A.

5.1.2 Singular Value Plot

Singular values of the matrix are plotted in the Figure 14.

Since all rows of Φ are independent none of the singular values are zero. First (maximum) SVD value is 111.25923 and the minimum is 3.2886 and the Condition number of the matrix is 33.8318.

5.1.3 SVD on Modified Matrix

After taking sum of elements in each row and replacing with the 400th column (this can be done for any column) we had done SVD on the Φ matrix. The U matrix obtained after decomposition is shown in the Figure 15. As the data is randomly spread we were not able to find the significant difference between the modified and unmodified U matrices. The V matrix obtained after decomposition is shown in Figure 16. In the V matrix one Eigen vector comes out higher than the other vectors, which is encircled inside the red circle, which directly point out increase in the one singular value and it came likewise also. Singular value plot for modified matrix is shown in the Figure 17. Here in modified case the first (maximum) singular value came around 526.6753 and the minimum

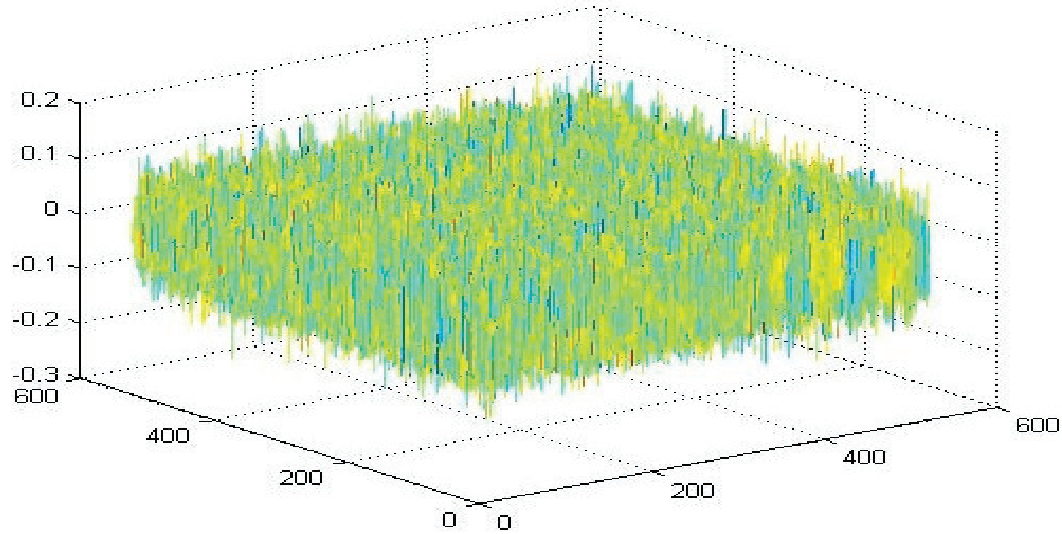

Figure 12. U matrix for an unmodified matrix Φ.

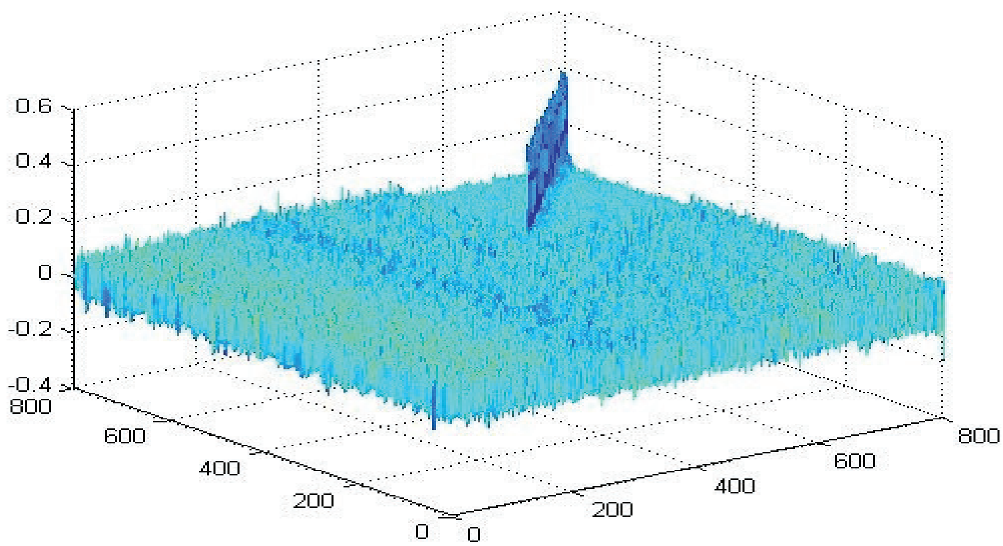

Figure 13. V matrix for an unmodified reconstruction matrix Φ.

singular value came as 0. 2.8478 and the condition number is 184.9411. Increase in condition number normally deteriorates the matrix, so we made the reconstruction matrix using the modified V and U unmodified SVD values. The reconstruction quality still preserved indicating that the mutual in coherence plays the key than the increased singular value. We had done SVD decomposition of the reconstruction matrix when an identity matrix is used as sparsifying matrix. Figure 18 shows the 2 dimensional plot of V matrix with and without modification. Instead of adding columns to the reconstruction matrix with unit scaling, smaller integer scaling also may be used (e.g. 2, 3 etc). A negative scaling also results in same improvement, indicating that the scaling can be done in both directions. Figure 19 shows the V matrix in that case. In V matrix the one element of Eigen vector with unit magnitude was found in the opposite direction. Two similarly scaled, one positive and the other negative also yielded same results.

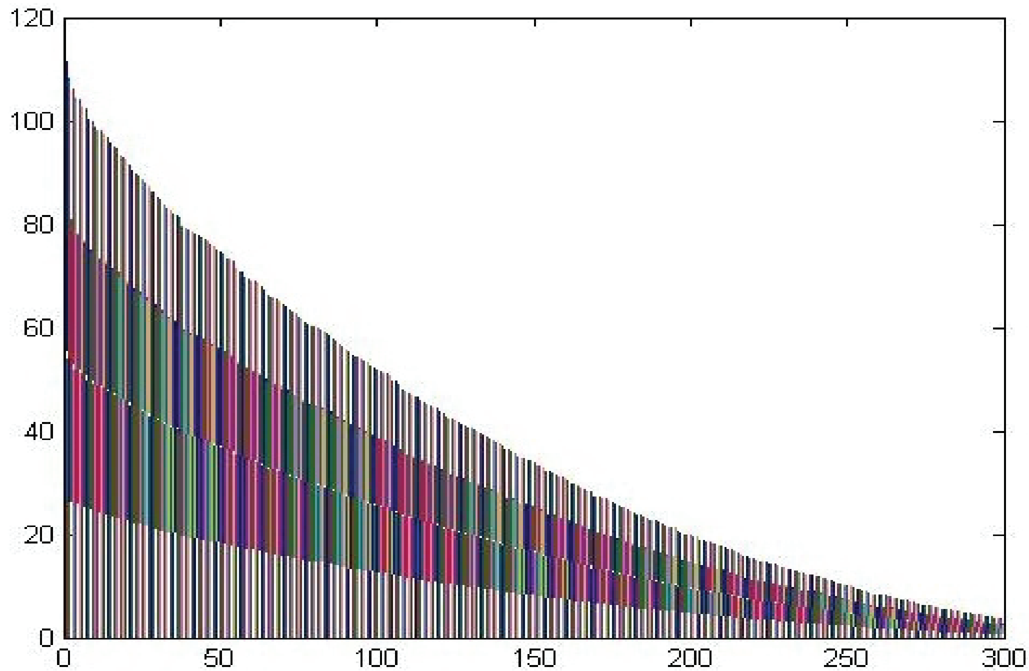

Figure 14. Singular values for the unmodified reconstruction matrix Φ.

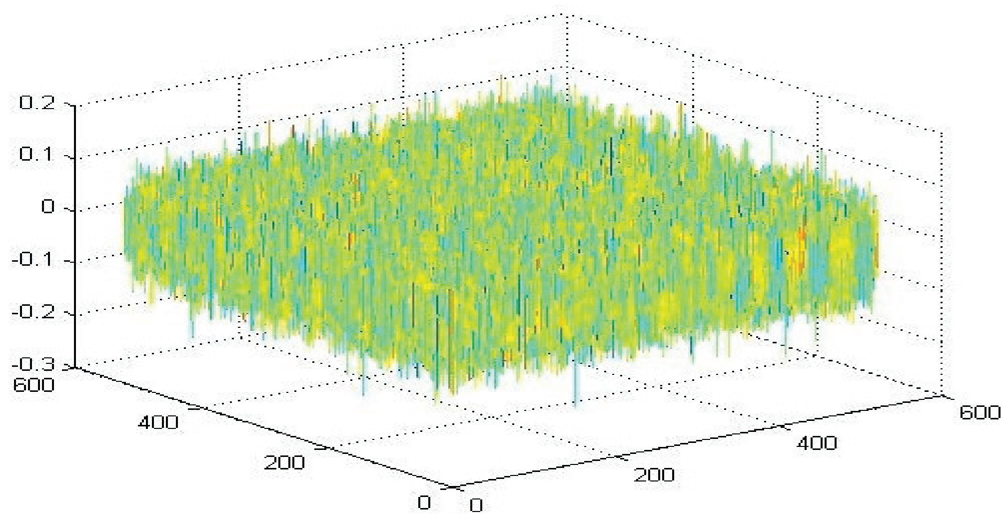

Figure 15. U matrix for the modified reconstruction matrix Φ.

5.2 QR Decomposition

QR decomposition factorize the matrix (A) into two matrices (Q and R) such that A = QR, where Q is an orthogonal matrix and R an upper triangular matrix.

5.2.1 QR Decomposition on the Unmodified Reconstruction Matrix

We had done the QR decomposition on the modified and unmodified reconstruction matrices and the results are plotted. Figure 20 and Figure 21 shows

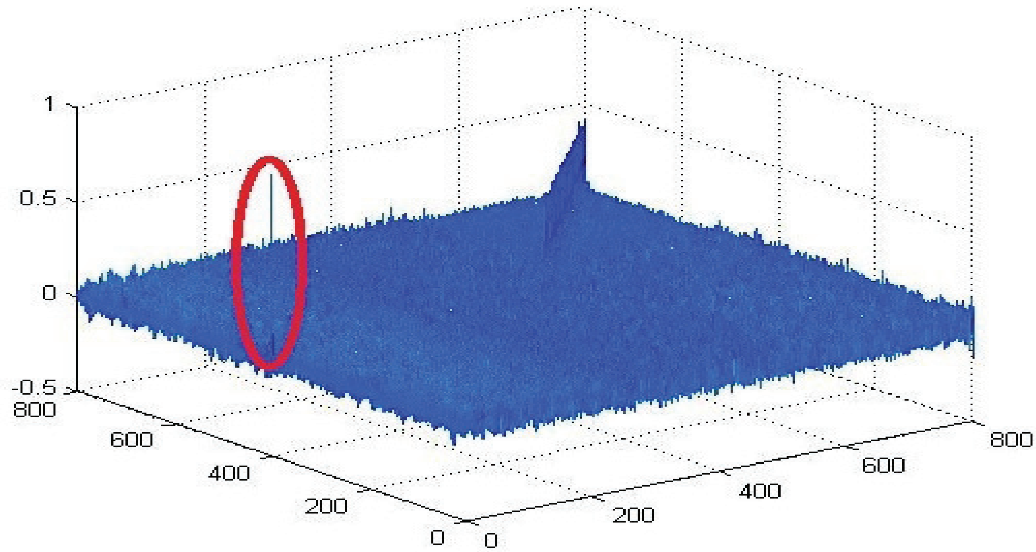

Figure 16. V matrix for the modified matrix.

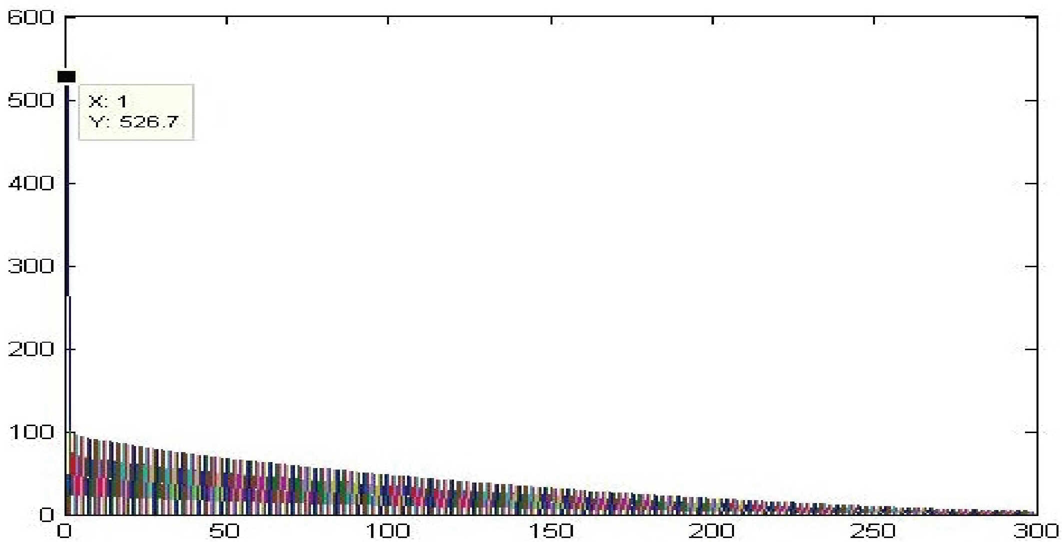

Figure 17. Singular value plot for the modified reconstruction matrix Φ. Notice that the first singuar value being very high compared to others.

the Q and R matrices of unmodified reconstruction matrix. Q matrix of unmodified reconstruction matrix. Q matrix came as a randomly distributed orthogonal matrix such that $QQ^T = 1$. R matrix obtained from the unmodified matrix after the reconstruction is shown in the Figure 21.

5.2.2 QR Decomposition on the Modified Reconstruction Matrix

The Q and R matrices obtained are shown in Figure 22 and Figure 23.

Q matrix for the modified reconstruction matrix obtained is shown in Figure 22. The data are randomly

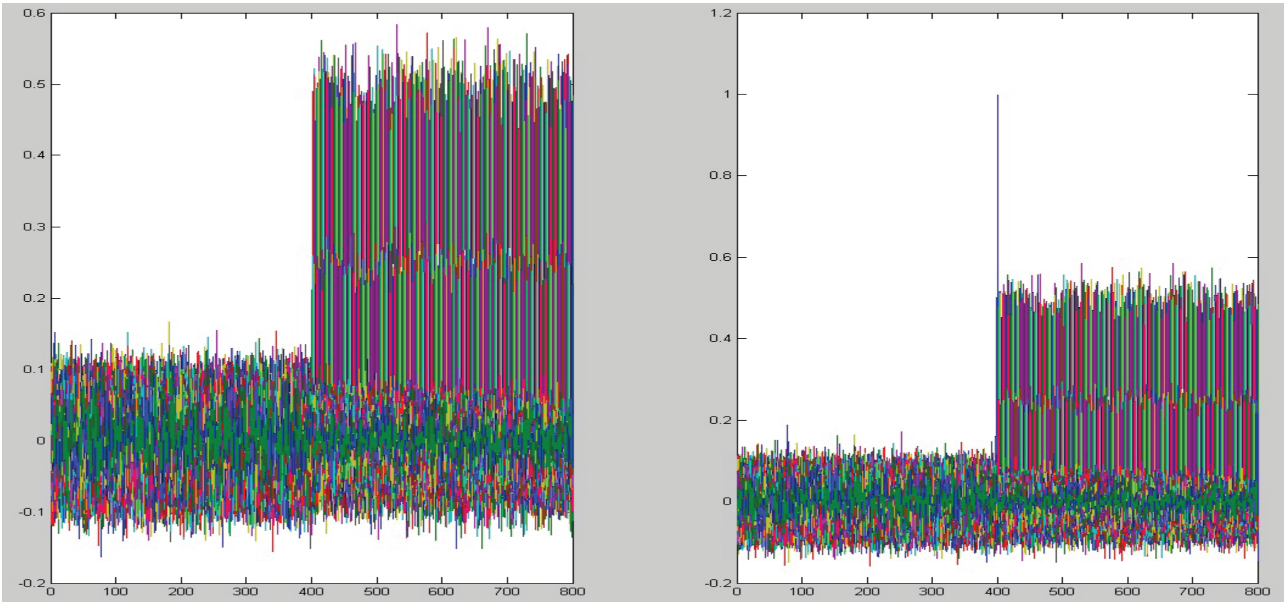

Figure 18. Diagonal elements of V Matrix with identity matrix as basis, without and with modification on reconstruction matrix,

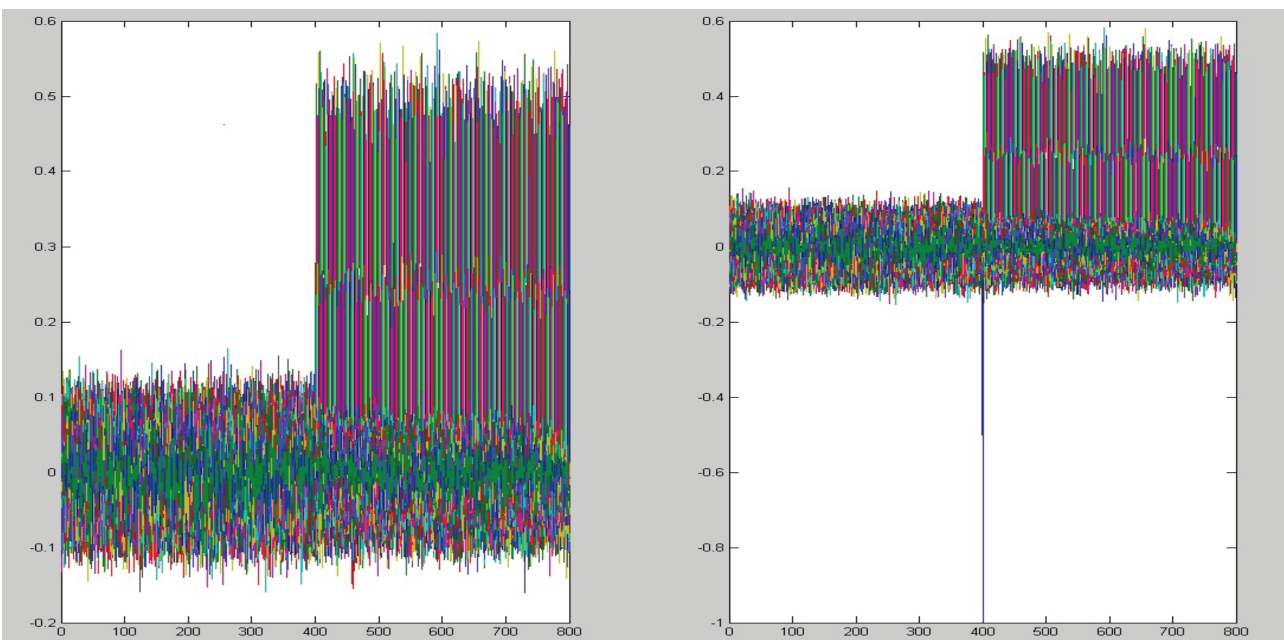

Figure 19. Diagonal elements of V matrix with identity matrix as basis, without and with modification.

distributed for this orthogonal matrix. R matrix for the modified Φ is shown in Figure 23. In R matrix there is set of high value elements through 400th column (where the row sum replaced the corresponding element).

6. Conclusion

It is a basic question in Compressed Sensing that how information content can be improved without affect-

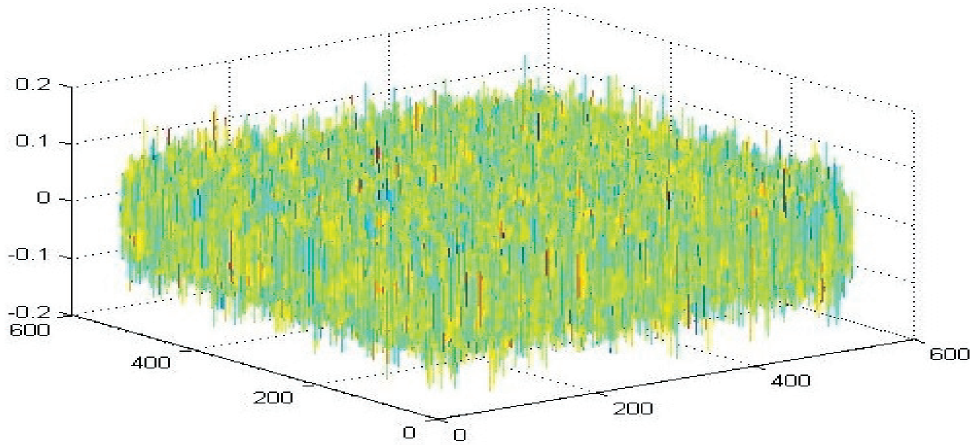

Figure 20. Q matrix of the unmodified matrix Φ.

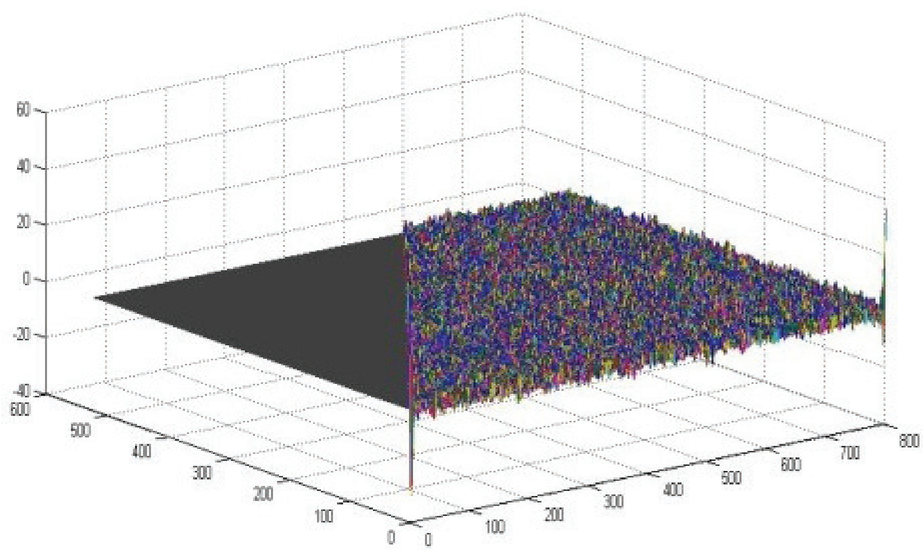

Figure 21. R matrix after QR for unmodified reconstruction matrix Φ.

ing the structure of the random matrix. Our work gives one such solution, which is evident from results and their analysis. By the changes we made on the random re-construction, one could conclude that we have generated a condition that mutual incoherence of one vector remained high while the solution is iterated.

However, the scaling had no effect. This is proved by taking the V matrix of the modified matrix and using the singular values of the un-modified matrix, also resulted in a similar improvement in reconstruction.

Normally, the singular values are affecting scale of the vectors in V and therefore the large value or the first singular value has no significant contribution in the reconstruction. To establish this, we have reconstructed the matrix with previous singular values and the V matrix corresponding to the modified matrix and it has been found that the quality of reconstruction is preserved. This is evident from a visualization of the V matrix, which changed only by one unit vector value contained in the matrix. Of course, some modest changes happened to

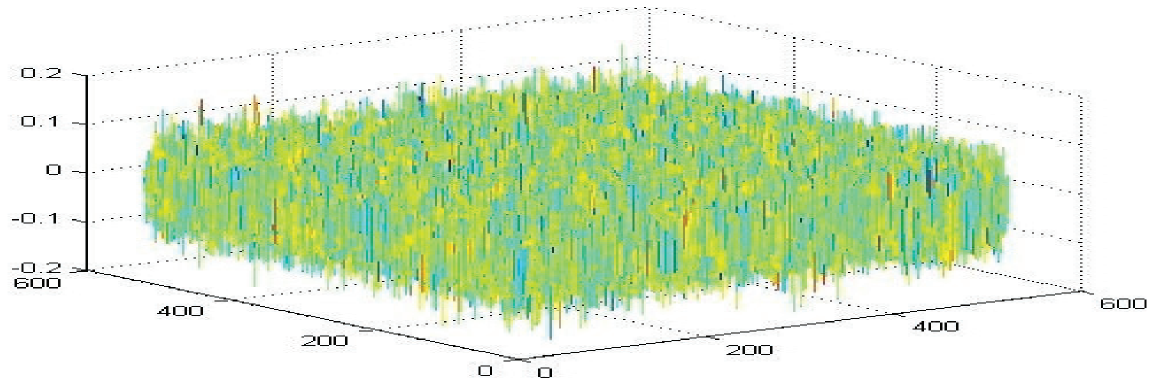

Figure 22. Q matrix for modified reconstruction matrix Φ.

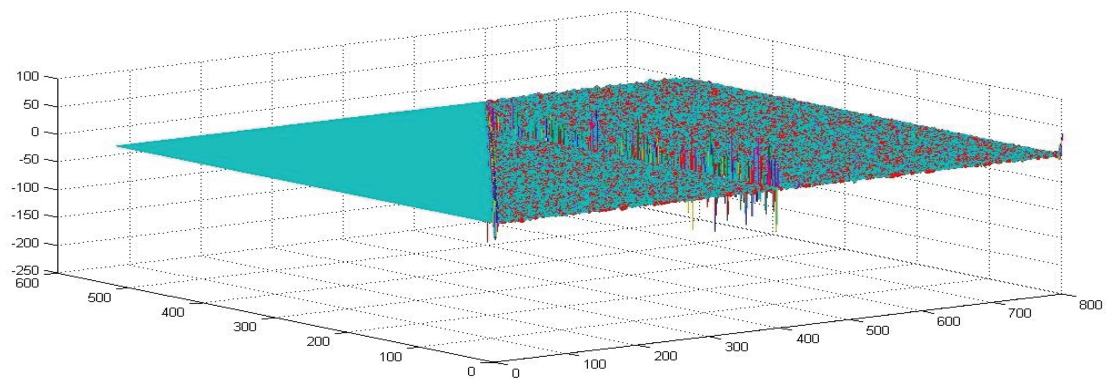

Figure 23. In R matrix there is set of high value elements through 400th column (where the row sum replaced the corresponding element).

other vectors, but it remained random unitary matrix in general.

The matrix Q remained as a random unitary matrix. This paper also explains in depth how E-Splines can be used in compressed sensing. However, Biorthogonal wavelets were also used for a comparison. The improvements in both cases are worth noting.

7. References

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