

Stability and Convergence of Spectral Approach for Fractional Integral-Differential Equation based on Legendre Basis

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Abstract

Background: In this study, we use operational Tau method (OTM) for finding the answer for fractional integral-differential equations (FIDEs). **Methods:** We prove that the approximated solutions of the Legendre Tau method converge to the exact solution in the L^2 norm. Also, some numerical findings are presented to clearly show the better performance of the proposed approach. **Results:** Outcomes reveals that the spectral approach based on the shifted Legendre basis can be considered as a structurally simple method that is typically applied for numerical solve of FIDEs. Also, our concentration restricted to linear Volterra FIDEs, we propose the approach to be developed to more common FIDEs. Despite the relatively low degrees utilized the numerical findings demonstrate the better performance of the spectral approach, in real condition, by considering the Legendre basis. **Conclusion:** Although the spectral rate of convergence illustrates the error of the Legendre spectral method demonstrates a tendency to increase fast.

Keywords: Integro-Differential Equations, Legendre Basis, Stability, Spectral Method

1. Introduction

It is said that several phenomena in various branches of science can be explained properly by patterns utilizing math instruments from fractional calculations, i.e., the approach of differential and integrals of fractional non-integer order. This mathematical event let describing an actual object more accurately¹.

Some kind of fractional boundary value problems (FBVPs) with Caputo's derivatives, and a coupled system of fractional differential equations have been evaluated by Kilbas and Trujillo², Zhang³, respectively. In⁴, they investigated the existence and uniqueness of solving of some types of FBVPs with Caputo's differentials and Riemann-Liouville differentials, respectively.

Besides, in these cited works, more contributions⁵ have been made to the analytical and numerical investigation of the solutions of FBVPs. Recently, multiple numerical

approaches to solve fractional integro-derivate equations (FIDEs) were given⁶.

Being and uniqueness of answers for the fractional integro differential equalizations in Banach are discussed in⁷. Wittayakiattilerd⁸ considered fractional integro-differential equations of mixed type with delay in the Riemann-Liouville sense. Local and global existence and uniqueness of mild solution are verified by utilizing a group of solution operators and the contraction mapping issues on Banach space.

The great aim of our study is to study the fractional integro-differential equations

$$D^\alpha y(x) = f(x) + \lambda \int_0^x k(x,t)y(t)dt, \quad (1)$$

subject to the initial values

$$y^{(i)}(0) = d_i, \quad i = 0, 1, \dots, m-1, \quad 1 \leq m = [\alpha], \quad m \in \mathbb{N}, \quad (2)$$

with $f \in L^2([0,1])$, $k \in L^2([0,1] \times [0,1])$ are given functions, $y(x)$ is the unknown function to be

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determined, D^α is the fractional derivative. In our paper, we let $\lambda = 1$.

Sudsutad⁹ presented some new Being and uniqueness results for FIDEs based on the Banach contraction principle and Krasnoselskii's fixed point theorem.

Vanani¹⁰ solved the Volterra fractional integro-differential equation by using operational Tau method (OTM) and presented an algorithm to finding approximate solution.

In this work, we investigate an approximate solution by applying operational spectral method and using shifted Legendre polynomials and then, analyzes the accuracy by presenting spectral rate.

2. Notation and Preliminaries

For $m \in \mathbb{N}$ to be the tiniest integral that is prominent than or equivalent to α , the Caputo's fractional acquired scoundrel of order $\alpha > 0$, is determined as:

$$D^\alpha y(x) = \begin{cases} I^{m-\alpha} D^m y(x) & m-1 < \alpha < m \\ D^m y(x) & \alpha = m \end{cases} \quad (3)$$

where

$$I^\nu y(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} y(t) dt, \quad \nu > 0, \quad x > 0.$$

For the Caputo's derivative [8]:

$$D^\alpha x^\beta = \begin{cases} 0, & \beta \in \{0,1,2,\dots\} \text{ and } \beta < m, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} & \beta \in \{0,1,2,\dots\} \text{ and } \beta \geq m. \end{cases} \quad (4)$$

Revive that for $\alpha \in \mathbb{N}$; the Caputo differential scoundrel matches with the general differential scoundrel of integral form. Comparable to integer-order differentiation, Caputo's fractional differentiation is a straight action:

$$D^\alpha (\lambda g(x) + \mu h(x)) = \lambda D^\alpha g(x) + \mu D^\alpha h(x), \quad (5)$$

where λ and μ are constants.

The Legendre polynomials $\{L_i(t); i = 0,1, \dots\}$ are defined on the interval $(-1,1)$. To use certain polynomials on the interval $(0,1)$, we determine the so-called moved Legendre polynomials by adding the correction of parameter $t=2x-1$. Let the moved Legendre polynomials $L_{-i}(2x-1)$ be expressed by $L_{1,i}(x)$, satisfying the orthogonality relation

$$\int_0^1 L_{1,i}(x)L_{1,j}(x)w_1(x)dx = \frac{1}{2i+1} \delta_{ij} := h_i, \quad (6)$$

where $w_1(x) = w(x) = 1$.

The analytic form of the shifted Legendre polynomial of degree n is given by

$$L_{1,i}(x) = \sum_{k=0}^i (-1)^{i+k} \frac{(i+k)! x^k}{(i-k)! (k!)^2}. \quad (7)$$

Theorem 1. The fractional acquired of series α in the Caputo mind for the moved Legendre polynomials is provided by

$$D^\alpha L_{1,i}(x) = \sum_{l=0}^{\infty} Z_\alpha(i,l) L_{1,l}(x), \quad i = m, m+1, \dots \quad (8)$$

where

$$Z_\alpha(i,l) = \sum_{k=m}^i (-1)^{i+k} \frac{(2l+1)(i+k)! \Gamma(k-l+\alpha+1)}{(i-k)! k! \Gamma(k-\alpha+1) \Gamma(k-\alpha+l+1)}. \quad (9)$$

3. A Shifted Legendre Tau Method

The primary purpose of this part is to state the construction of OTM and its relevance to the fractional calculus.

The main idea of OTM is that we attempt a polynomial to fair $y(x) \in L^2 [0,1]$ where

$L^2 [0,1]$ is the period of all functions $f: [0,1] \rightarrow \mathbb{R}$, with $\|f\|^{L^2} < \infty$ with

$$\|f\|^2 = \langle f, f \rangle = \int_0^1 f^2(x) dx.$$

For simplicity, we let $I = [0,1]$.

Consider first introduce some basic notation that will be utilized in the sequel. We set

$$S_N(I) = \text{span}\{L_{1,0}(x), L_{1,1}(x), \dots, L_{1,N}(x)\}.$$

The moved Legendre-Tau approach to Eq. (1) subject to Eq. (2) is to get $y_N(x) \in S_N(I)$ such that

$$\langle D^\alpha y_N, L_{1,k} \rangle = \langle f, L_{1,k} \rangle + \langle \int_0^1 k(\cdot,s) y_N(s) ds, L_{1,k} \rangle, \quad (10)$$

and

$$y_N^{(i)}(0) = d_i, \quad i = 0,1, \dots, m-1. \quad (11)$$

Here, the central concept is that we use a truncated set of moved Legendre polynomials to approaching the unfamiliar function, and the fractional-differential director of this truncated series is developed by moved Legendre polynomials themselves (Theorem 1), and later

the coefficients of this group are taken to be identical to the coefficients of the right-hand side development. We express

$$y_N(x) = \sum_{j=0}^N a_j L_{1,j}(x), \quad \mathbf{a} = (a_0, a_1, \dots, a_N)^T, \quad N > m,$$

$$f_k = \langle f, L_{1,k} \rangle, \quad k = 0, 1, \dots, N - m, \quad (12)$$

$$\mathbf{f} = (f_0, f_1, \dots, f_{N-m}, d_0, d_1, \dots, d_{m-1})^T,$$

then Eq. (10) can be written as

$$\sum_{j=0}^N a_j \langle D^\alpha L_{1,j}, L_{1,k} \rangle = f_k + \sum_{j=0}^N a_j \langle \int_0^{\cdot} k(\cdot, s) L_{1,j}(s) ds, L_{1,k} \rangle$$

By using **Theorem 1** and approximating $D^\alpha L_{1,j}(x) \cong \sum_{i=0}^N Z_\alpha(i, l) L_{1,i}(x)$, we get

$$\sum_{j=0}^N a_j \left(\sum_{i=0}^N Z_\alpha(j, i) \langle L_{1,i}, L_{1,k} \rangle - \langle \int_0^{\cdot} k(\cdot, s) L_{1,j}(s) ds, L_{1,k} \rangle \right) \cong f_k, \quad (13)$$

On the other hand, according to Eq. (5) and using equality instead of, we have

$$\sum_{j=0}^N a_j \left(Z_\alpha(j, k) h_k - \langle \int_0^{\cdot} k(\cdot, s) L_{1,j}(s) ds, L_{1,k} \rangle \right) = f_k. \quad (14)$$

The initial conditions are converted to

$$\sum_{j=0}^N a_j L_{1,j}^{(i)}(0) = d_i, \quad i = N - m + 1, \dots, N. \quad (15)$$

We define the square matrix $\mathbf{A} = (a_{kj})_{0 \leq j, k \leq N}$ with

$$a_{jk} = \left(Z_\alpha(j, k) h_k - \langle \int_0^{\cdot} k(\cdot, s) L_{1,j}(s) ds, L_{1,k} \rangle \right). \quad (16)$$

Note that, the second term of a_{jk} is calculated as follows:

Let $\{x_l, w_l\}_{l=0}^N$ be Legendre-Gauss quadrature nodes and weights (Theorem 3.29). A first approximation to the second term using a Legendre collocation approach is

$$\langle \int_0^{\cdot} k(\cdot, s) L_{1,j}(s) ds, L_{1,k} \rangle = \int_0^1 \left(\int_0^x k(x, s) L_{1,j}(s) ds \right) L_{1,k}(x) dx$$

$$= \sum_{l=0}^N w_l \left(\int_0^{x_l} k(x_l, s) L_{1,j}(s) ds \right) L_{1,k}(x_l). \quad (17)$$

However, the integral term in Eq. (17) cannot be assessed correctly, we convert the integral interval $[0, x_l]$ to $[0, 1]$ and apply a Gaussian model quadrature dominion to evaluate the integral. More specifically, under the direct transformation

$$s := s^{(i)} = x_l \theta,$$

$$\theta := \theta^{(i)} = \frac{s}{x_l}, \quad \theta \in I, s \in [0, x_l], \quad (18)$$

the Eq. (17) becomes

$$\langle \int_0^{\cdot} k(\cdot, s) L_{1,j}(s) ds, L_{1,k} \rangle = \sum_{l=0}^N w_l \left(\int_0^{x_l} k(x_l, s) L_{1,j}(s) ds \right) L_{1,k}(x_l)$$

$$= \sum_{l=0}^N w_l \left(x_l \int_0^1 k(x_l, x_l \theta) L_{1,j}(x_l \theta) d\theta \right) L_{1,k}(x_l).$$

Again, by using Gaussian type quadrature rule, we can obtain the value of the second term of Eq. (16).

Therefore, Eq. (14) and Eq. (15) are equivalent to the matrix equation

$$\mathbf{Aa} = \mathbf{f}.$$

Solving the above system yields the unknown vector $\mathbf{a} = (a_0, a_1, \dots, a_N)^T$ and then we obtain an approximate solution of FIDES.

4. Accuracy of Solution

Let $H^m(I)$ denotes the Sobolev space of consisting all functions $y(x)$ on I such that $y(x)$ and all its weak derivatives up to order m are in $L^2(I)$. The norm and semi norm of $H^m(I)$ are defined respectively by

$$\|y(x)\|_{H^m(I)}^2 = \sum_{k=0}^m \left\| \frac{\partial^k}{\partial t^k} y(x) \right\|_{L^2(I)}^2, \quad (19)$$

and

$$|y|_{H^{m,N}(I)}^2 = \sum_{k=\min(m,N)}^N \|y^{(k)}\|_{L^2(I)}^2. \quad (20)$$

Let $P_N(I)$ be of polynomials with degree $\leq N$ on I and p_N be the orthogonal projective operator $L^2(I)$

onto $P_N(I)$. Then, for any function f in $L^2(I)$, $p_N f$ belongs to $P_N(I)$ and satisfies

$$\int_0^1 (f - p_N f)(t)g(t)dt = 0, \quad \forall g \in P_N(I).$$

To prove the convergence of our method, we consider the following assumptions:

$$(A_1) y(x) \in H^k(I),$$

$$(A_2) f(x) \in C(I),$$

$$(A_3) k(x, s), \partial_x k(x, s) \in L^\infty(D) \text{ where } D = \{(x, s): 0 \leq s \leq x \leq 1\},$$

(A₆) From [26], the following relations with shifted Legendre polynomials and shifted Legendre-Gauss-Lobatto nodal points for $k \geq 1$ may readily be obtained as

$$\|y - p_N(y)\|_{H^{k:N}(I)} \leq C_1 N^{2l - \frac{1}{2} - k} |y|_{H^{k:N}(I)}, \quad (21)$$

$$\|I_N(y) - y\|_{L(I)} \leq C_2 N^{-k} |y|_{H^{k:N}(I)}, \quad (22)$$

$$\|\partial_x(I_N(y) - y)\|_{L^2(I)} + N \|I_N(y) - y\|_{H^{k:N}(I)} \leq ((N - m + 2)(N + m))^{\frac{(1-m)}{2}} \|\partial_x^m y\|_{H^{k:N}(I)}, \quad (23)$$

where $y \in H^k(I)$, and C_1, C_2 and C_3 are constants independent of N .

In the following theorem, an error estimation for the approximate solution of Eq. (1) with supplementary conditions of Eq. (2) is calculated. Consider $e_N(x) = y(x) - y_N(x)$, be the error function of the evaluation solution $y_N(x)$.

Theorem 1. Assume $f: I \rightarrow \mathbb{R}$ and $k: I \times I \rightarrow \mathbb{R}$ satisfy the assumptions (A₂) and (A₃), respectively. Also, assume that the correct solution $y(x)$ satisfies the assumption (A₁). Let y_N be the approximation solution of Eq. (1) is given by spectral method with shifted Legendre polynomials. Then

$$\|e_{y_N}\|_{L^2(I)} \leq C_1 C^* N^{-1} \left(C_2 N^{\frac{5}{2} - k} \|y\|_{H^{k:N}(I)} + \|y\|_{H^{k:N}(I)} \right) + C_f N^{-1} |f|_{H^{k:N}(I)} + C_3 N^{-1} \left(\max_{x \in I} |k(x, x)| + \max_D |\partial_x k| \right) \|y\|_{H^{k:N}(I)}. \quad (24)$$

Proof. Applying the interpolation operator on both sides of Eq. (1) and using the linearity of it, we get

$$I_N(D^\alpha y_N(x)) = I_N(f(x)) + I_N\left(\int_0^x k(x, t)y_N(t)dt\right).$$

According to Eq. (3), we get

$$\frac{1}{\Gamma(m - \alpha)} I_N\left(\int_0^x (x - s)^{m - \alpha - 1} y_N^{(m)}(s) ds\right) = I_N(f(x)) + I_N\left(\int_0^x k(x, t)y_N(t)dt\right). \quad (25)$$

By subtracting Eq. (25) from Eq. (1), the following equation is obtained

$$D^\alpha y(x) - I_N(D^\alpha y_N(x)) = f(x) - I_N(f(x)) + \int_0^x k(x, t)y(t)dt - I_N\left(\int_0^x k(x, t)y_N(t)dt\right). \quad (26)$$

With the aid of the definition of Caputo fractional derivative in Eq. (3), we have

$$e_{y_N}(x) = \frac{1}{\Gamma(m - \alpha)} \left(\int_0^x (x - s)^{m - \alpha - 1} y^{(m)}(s) ds - I_N\left(\int_0^x (x - s)^{m - \alpha - 1} y_N^{(m)}(s) ds\right) \right) - (f(x) - I_N(f(x))) - \left(\int_0^x k(x, t)y(t)dt - I_N\left(\int_0^x k(x, t)y_N(t)dt\right) \right) = e_{I_N} - e_f(x) - e_{k_y}(x), \quad (27)$$

where

$$e_{I_N} = \frac{1}{\Gamma(m - \alpha)} \left(\int_0^x (x - s)^{m - \alpha - 1} y^{(m)}(s) ds - I_N\left(\int_0^x (x - s)^{m - \alpha - 1} y_N^{(m)}(s) ds\right) \right),$$

$$e_f(x) = f(x) - I_N(f(x)),$$

and

$$e_{k_y}(x) = \int_0^x k(x, t)y(t)dt - I_N\left(\int_0^x k(x, t)y_N(t)dt\right).$$

By virtue of Eq. (22), there exists $C_f > 0$ such that

$$\|e_f\|_{L^2(I)} \leq C_f N^{-1} |f|_{H^{k:N}(I)}. \quad (28)$$

By using Eq. (21), $e_{I_N}(x)$ satisfies the following inequality

$$\|e_{I_N}\|_{L^2(I)} \leq C_1 N^{-1} \|D_*^\alpha y_N\|_{H^{k:N}(I)}, \quad (29)$$

where

$$D_*^\alpha y_N(x) = \int_0^x (x - s)^{m - \alpha - 1} y_N^{(m)}(s) ds.$$

On the other hand, linear operator $D_*^\alpha: \mathbb{P}_N \rightarrow \mathbb{P}_N$ is continuous and bounded, then there exists C^* such that

$$\|D_*^\alpha y_N\|_{H^{k:N}(I)} \leq C^* \|y_N\|_{H^{k:N}(I)}. \quad (30)$$

Therefore, by applying Eq. (29) and Eq. (30) and then using the definition of $e_N(x)$, we infer

$$\|e_{I_N}\|_{L^2(I)} \leq C_1 C^* N^{-1} \|y_N\|_{H^{k:N}(I)}$$

$$\leq C_1 C^* N^{-1} (\|e_N\|_{H_W^{1:N}(I)} + \|y\|_{H_W^{1:N}(I)}). \quad (31)$$

From Eq. (21), there exists $C_2 > 0$ such that

$$\|e_N\|_{H_W^{1:N}(I)} \leq C_2 N^{\frac{3}{2}-k} \|y\|_{H_W^{k:N}(I)}. \quad (32)$$

Finally, from Eq. (31) and Eq. (32), we have

$$\|e_{I_N}\|_{L^2(I)} \leq C_1 C^* N^{-1} (C_2 N^{\frac{3}{2}-k} \|y\|_{H_W^{k:N}(I)} + \|y\|_{H_W^{1:N}(I)}). \quad (33)$$

The rest of our proof is proving boundedness of e_{KY} . According to Eq. (23), for $k = 1$, there exists $C_3 > 0$ such that we will have

$$\begin{aligned} \|e_{KY}\|_{L^2(I)} &\leq C_3 N^{-1} \left\| k(x,x)y(x) + \int_0^x \partial_x k(x,s)y(s)ds \right\| \\ &\leq C_3 N^{-1} \left(\max_{x \in I} |k(x,x)| + \max_D \|\partial_x k\|_\infty \right) \|y\|_{H_W^{1:N}(I)}. \end{aligned} \quad (34)$$

Finally, the three inequalities (33), (34) and (23) yield

$$\begin{aligned} \|e_{y_N}\|_{L^2(I)} &\leq C_1 C^* N^{-1} (C_2 N^{\frac{3}{2}-k} \|y\|_{H_W^{k:N}(I)} + \|y\|_{H_W^{1:N}(I)}) + C_f N^{-1} \|f\|_{H_W^{k:N}(I)} \\ &\quad + C_3 N^{-1} \left(\max_{x \in I} |k(x,x)| + \max_D \|\partial_x k\|_\infty \right) \|y\|_{H_W^{1:N}(I)}. \end{aligned}$$

5. Numerical Illustrations

To dispense the efficiency of the scheme, we perform it to solve three cases.

Example1. Suppose the below integro differential equation [10]

$$D^{0.75} y(x) = -\left(\frac{e^x x^2}{5}\right) y(x) + \frac{6x^{2.25}}{3.25} + \int_0^x e^x t y(t) dt,$$

With the initial condition $y(0) = 0$, the correct solution $y(x) = x^3$.

We use the proposed approach with $N = 6$, and we evaluate solution as:

$$y_N(x) = \sum_{j=0}^N a_j L_{1,j}(x).$$

Thus, by solving the corresponding linear system of this problem, we get

$$\begin{aligned} y_N(x) &= 7.2 \times 10^{-21} - 2.0 \times 10^{-19} x + 1.1 \times 10^{-18} x^2 + 0.9999999999 x^3 + 0.96192 \\ &\quad \times 10^{-19} x^4 - 0.3524608 \times 10^{-17} + 0.2369536 \times 10^{-17} \cong x^3. \end{aligned}$$

Example2. Consider the fractional integro-differential equation

$$y^{0.5}(x) = (\cos x - \sin x)y(x) + f(x) + \int_0^x x \sin t y(t) dt,$$

subject to initial condition $y(0) = 0$, and choose $f(x)$ so that the correct solve is $y(x) = x^2 + x$.

Our approximate solution for this integro-differential equation, using $N = 3$, is:

$$0.4 * 10^{-14} + 1.00032012400449 t + 0.999724628611392 t^2 \cong t^2 + t.$$

Example3. Suppose the below linear fourth-order fractional integro-differential

$$D^\alpha y(x) = x(1 + e^x) + 3e^x + y(x) - \int_0^x y(t) dt, \quad 0 < x < 1, \quad 3 < \alpha \leq 4,$$

with boundary conditions

$$y(0) = 1, \quad y''(0) = 2, \quad y(1) = e + 1, \quad y''(1) = 3e.$$

This problem has the exact solution $y(x) = 1 + x e^x$ in the case of $\alpha = 4$. In this example, we implement the Tau Legendre method for $\alpha = 3.25$ with $N = 20$ and $\alpha = 3.75$ with $N = 10$.

Our results for these values of N , have been shown in Figures 1-3. In the Tables 1 and 2, the numerical results of our proposed method are compared with those obtained by ADM in, FDTM considered in ¹¹ and Chebyshev-pseudo spectral method presented in ¹². We observe that the Tau approximation is an extremely good approximation to solution of FIDEs and is much better than the mentioned methods.

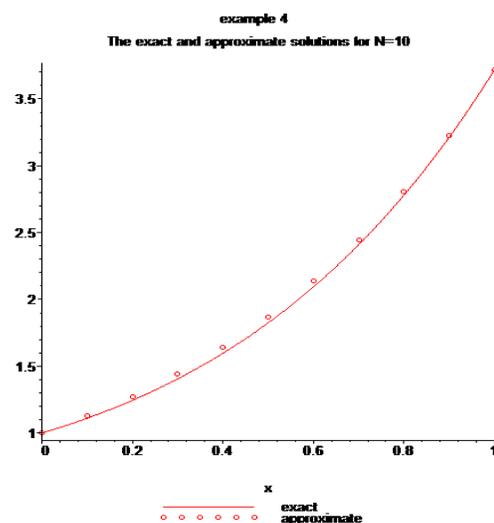


Figure 1. Describe the exact solution and approximate solution with $\alpha = 4$ for $N = 10$.

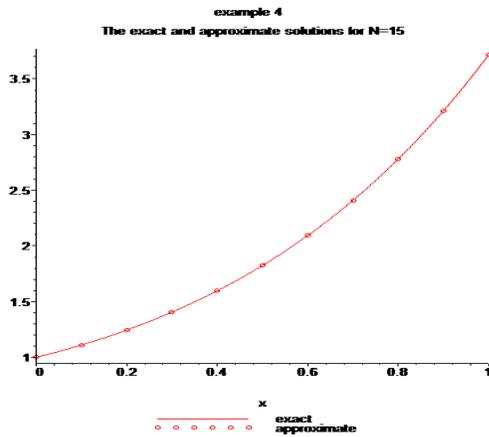


Figure 2. Describe the exact solution and approximate solution with $\alpha = 4$ for $N = 15$.

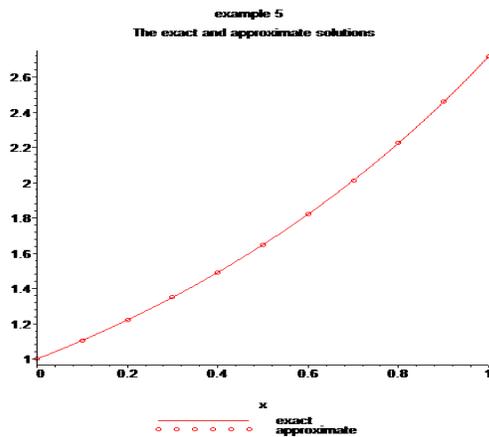


Figure 3. Describe the exact solution and approximate solution with $\alpha = 4$ for $N = 15$.

Table 1. Comparison of $y(t)$ for $\alpha = 3.25$ and $\alpha = 3.75$ for example 4

t	Method in	Our method with $\alpha = 3.25$	t	Method in	Our method with $\alpha = 3.75$
0.0	1.00000000	1.00000000	0.0	1.00000000	1.00000000
0.1	1.12713251	1.10402221	0.1	1.11574165	1.10726965
0.2	1.27412871	1.24400051	0.2	1.25407050	1.24414053
0.3	1.44419228	1.40495333	0.3	1.41822408	1.40495548
0.4	1.64216876	1.59552240	0.4	1.61210580	1.59612614
0.5	1.86980974	1.82003001	0.5	1.84034522	1.82219532
0.6	2.13679076	2.09321156	0.6	2.10836127	2.09324142
0.7	2.44531829	2.40854621	0.7	2.42242223	2.40908655
0.8	2.80675219	2.78010074	0.8	2.78973070	2.78026674
0.9	3.22745248	3.21324015	0.9	3.21854010	3.21344147
1.0	3.71828183	3.71828183	1.0	3.71828183	3.71828183

Table 2. Comparison of $y(t)$ for $\alpha = 3.25$ and $\alpha = 3.75$ and $\alpha = 4$, respectively for example 5

t	Method in	Our method with $\alpha = 3.25$	t	Method in	Our method with $\alpha = 3.75$
0.0	1.00000000	1.00000000	0.0	1.00000000	1.00000000
0.1	1.10093503	1.10757565	0.1	1.10659794	1.10593045
0.2	1.21465413	1.22583824	0.2	1.22402473	1.22281325
0.3	1.34025419	1.35562954	0.3	1.35333811	1.35174095
0.4	1.48271885	1.49842125	0.4	1.49577522	1.49396243
0.5	1.63838198	1.65534854	0.5	1.65274868	1.65089025
0.6	1.81382749	1.82825789	0.6	1.82585802	1.82411002
0.7	2.00612089	2.01867254	0.7	2.01687273	2.01539021
0.8	2.22060959	2.22898695	0.8	2.22777938	2.22669587
0.9	2.45675040	2.46136754	0.9	2.46077210	2.46019578
1.0	2.71828183	2.71828065	1.0	2.71828183	2.71828127

6. Conclusion

Our conclusions indicate that the spectral method based on the shifted Legendre basis can regard as a structurally simple algorithm that is conventionally applicable as numerical solution of FIDEs. Furthermore, we have limited our consideration to linear Volterra FIDEs; we demand the way to extend quickly to more customary FIDEs.

Despite the relatively small degrees used the numerical results show the excellent execution of the spectral approach, individually, with the Legendre base. Nevertheless, the alarming rate of convergence represents the failure of the Legendre spectral approach displays a trend to develop quickly.

7. Reference

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