Indian Journal of Science and Technology, *Vol 9(31), DOI: 10.17485/ijst/2016/v9i31/84442, August 2016*
ISSN (Print) : 0974-6846
ISSN (Online) : 0974-5645

Order Five Block Method for the Solution of Second Order Boundary Value Problems

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Abstract

Background/Objectives: Numerically solving second order Boundary Value Problems (BVPs) directly is an area that has drawn much attention in recent literature and hence the general aim of this article. **Methods/Statistical Analysis**: Block methods are developed using Taylor series approach and are applied to directly approximate the considered second order Boundary Value Problems. Certain specifications of the block method developed involves it being of step number k = 4 and order p = 5. This block method consists of mathematical expressions that will simultaneously provide results at the grid points and derivatives of the grid points for the k-step method. **Findings:** After the development of the new block method in this article, certain Boundary Value Problems are considered and the new block method is adopted to solve it numerically and comparison was made with their exact solution. The results show the new block method show superiority in comparison to other numerical methods of same order p = 5 adopted by other authors in previous literature. Hence, this new block method can be considered a more suitable numerical approach to approximate the numerical solutions of second order BVPs. **Applications/Improvements:** This block method is suitable for application in solving numerically second order BVPs with closer accuracy to the exact or analytic solutions of the differential equations under consideration.

Keywords: Block Method, Boundary Value Problems, Four-Step, Order Five, Second Order

1. Introduction

Boundary Value Problems are seen to have a wide range of application in the area of mathematical modelling. Some of these applications include modelling of chemical reactions, deflection and deformation of beams, heat power transmission, temperature distribution across a rod, boundary layer theory, amongst many others[1,2].

Due to the importance of Boundary Value Problems in real life situations, the need to find appropriate methods for approximating its solution is expedient. However, similar to the case of initial value problems, Boundary Value Problems sometimes have more than one solution or the solution may not exist. Hence numerical solutions are adopted to find an approximate solution to these Boundary Value Problems.

Quite a number of scholars have proposed numerical and approximate methods for solving Boundary Value Problems. However, the focus of this work is the two-point second order boundary value problem of the form,

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) \tag{1}$$

with Dirichlet boundary conditions,

$$y(a) = \alpha\,,\ y(b) = \beta\,.$$

Some authors who have discussed finding approximate solutions to (1) above include[3,4] who proposed an extension to the conventional Adomian decomposition method, [5] who adopted cubic hermite collocation method, [6,7] who developed block methods using direct integration and collocation approaches, [2] using collocation method with Haar wavelets, amongst many others.

However, this work presents a k-step method ($k = 4$) to numerically approximate problems in the form (1) above and the a comparison of the numerical results was made with the results from the work of[8] who

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used linear multistep technique having power series as the basis function to develop the block methods to solve (1) and [9] who adopted a direct three-point block one-step method, also to solve (1). The work of these authors were chosen for comparison because the methods proposed in their individual works is also of order p = 5 which gives a good foundation for comparison with the method proposed in this work.

The other sections of this paper will discuss the following: Section 2 will present the methodology, Section 3 will show and discuss the results to the numerical problems considered and finally Section 4 concludes this paper.

2. Methodology

The section gives a detailed description of how the method is derived. Hence, the first step entails the derivation of the k-step discrete scheme and the derivatives needed to obtain the needed mathematical expressions.

Consider the following expression for deriving the $k = 4$ discrete scheme.

$$y_{n+4} = \alpha_0 y_n + \alpha_1 y_{n+1} + \sum_{j=0}^{k} \beta_j \zeta_{n+j} \tag{2}$$

Equation (2) above can equally be expressed as:

$$y_{n+4} = \alpha_0 y_n + \alpha_1 y_{n+1} + \big(\beta_0 \zeta_n + \beta_1 \zeta_{n+1} + \beta_2 \zeta_{n+2} + \beta_3 \zeta_{n+3} + \beta_4 \zeta_{n+4}\big) \tag{3}$$

Now, expanding y_{n+4}, y_n, y_{n+1}, ζ_n, ζ_{n+1}, ζ_{n+2}, ζ_{n+3} and ζ_{n+4} using Taylor series expansion and substituting back in (3) give the following matrix representation $Ax = B$ as:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{h^2}{2!} & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{h^3}{3!} & 0 & h & 2h & 3h & 4h \\ 0 & \frac{h^4}{4!} & 0 & \frac{h^2}{2!} & \frac{(2h)^2}{2!} & \frac{(3h)^2}{2!} & \frac{(4h)^2}{2!} \\ 0 & \frac{h^5}{5!} & 0 & \frac{h^3}{3!} & \frac{(2h)^3}{3!} & \frac{(3h)^3}{3!} & \frac{(4h)^3}{3!} \\ 0 & \frac{h^6}{6!} & 0 & \frac{h^4}{4!} & \frac{(2h)^4}{4!} & \frac{(3h)^4}{4!} & \frac{(4h)^4}{4!} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4h \\ \frac{(4h)^2}{2!} \\ \frac{(4h)^3}{3!} \\ \frac{(4h)^4}{4!} \\ \frac{(4h)^5}{5!} \\ \frac{(4h)^6}{6!} \end{pmatrix}$$

Using the formula $x = A^{-1}B$, the following values are obtained:

$$(\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4)^T = \left(-3, 4, \frac{9h^2}{40}, \frac{83h^2}{30}, \frac{37h^2}{20}, \frac{11h^2}{10}, \frac{7h^2}{120}\right)^T \tag{4}$$

which further produces the following method upon substitution back in (3),

$$y_{n+4} = -3y_n + 4y_{n+1} + \frac{h^2}{120}(27\zeta_n + 332\zeta_{n+1} + 222\zeta_{n+2} + 132\zeta_{n+3} + 7\zeta_{n+4}) \tag{5}$$

The additional methods for the discrete scheme and their corresponding derivatives are obtained by following the same approach of deriving the discrete scheme (5). The required methods are obtained as follows:

$$\begin{aligned}
y_{n+2} &= -y_n + 2y_{n+1} + \frac{h^2}{240}(19\zeta_n + 204\zeta_{n+1} + 14\zeta_{n+2} + 4\zeta_{n+3} - \zeta_{n+4}), \\
y_{n+3} &= -2y_n + 3y_{n+1} + \frac{h^2}{240}(37\zeta_n + 432\zeta_{n+1} + 222\zeta_{n+2} + 32\zeta_{n+3} - 3\zeta_{n+4}), \\
y'_n &= \frac{1}{h}(-y_n + y_{n+1}) + \frac{h}{360}(-367\zeta_n - 540\zeta_{n+1} + 282\zeta_{n+2} - 116\zeta_{n+3} + 21\zeta_{n+4}), \\
y'_{n+1} &= \frac{1}{h}(-y_n + y_{n+1}) + \frac{h}{1440}(135\zeta_n + 752\zeta_{n+1} - 246\zeta_{n+2} + 96\zeta_{n+3} - 17\zeta_{n+4}), \\
y'_{n+2} &= \frac{1}{h}(-y_n + y_{n+1}) + \frac{h}{1440}(97\zeta_n + 1444\zeta_{n+1} + 666\zeta_{n+2} - 52\zeta_{n+3} + 5\zeta_{n+4}), \\
y'_{n+3} &= \frac{1}{h}(-y_n + y_{n+1}) + \frac{h}{1440}(119\zeta_n + 1296\zeta_{n+1} + 1578\zeta_{n+2} + 640\zeta_{n+3} - 33\zeta_{n+4}), \\
y'_{n+4} &= \frac{1}{h}(-y_n + y_{n+1}) + \frac{h}{1440}(81\zeta_n + 1508\zeta_{n+1} + 1050\zeta_{n+2} + 1932\zeta_{n+3} + 469\zeta_{n+4})
\end{aligned} \tag{6}$$

Putting together Equations (5) and (6), a matrix representation $Ax = B$ can also be given, where $x = \left(y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y'_{n+1}, y'_{n+2}, y'_{n+3}, y'_{n+4}\right)$.

Using the formula $x = A^{-1}B$ again, the following expressions are obtained:

$$\begin{aligned}
y_{n+1} &= y_n + hy'_n + \frac{h^2}{1440}(367\zeta_n + 540\zeta_{n+1} - 282\zeta_{n+2} + 116\zeta_{n+3} - 21\zeta_{n+4}), \\
y_{n+2} &= y_n + 2hy'_n + \frac{h^2}{90}(53\zeta_n + 144\zeta_{n+1} - 30\zeta_{n+2} + 16\zeta_{n+3} - 3\zeta_{n+4}), \\
y_{n+3} &= y_n + 3hy'_n + \frac{h^2}{160}(147\zeta_n + 468\zeta_{n+1} + 54\zeta_{n+2} + 60\zeta_{n+3} - 9\zeta_{n+4}), \\
y_{n+4} &= y_n + 4hy'_n + \frac{h^2}{45}(56\zeta_n + 192\zeta_{n+1} + 48\zeta_{n+2} + 64\zeta_{n+3}), \\
y'_{n+1} &= y'_n + \frac{h}{720}(251\zeta_n + 646\zeta_{n+1} - 264\zeta_{n+2} + 106\zeta_{n+3} - 19\zeta_{n+4}), \\
y'_{n+2} &= y'_n + \frac{h}{90}(29\zeta_n + 124\zeta_{n+1} + 24\zeta_{n+2} + 4\zeta_{n+3} - \zeta_{n+4}), \\
y'_{n+3} &= y'_n + \frac{h}{80}(27\zeta_n + 102\zeta_{n+1} + 72\zeta_{n+2} + 42\zeta_{n+3} - 3\zeta_{n+4}), \\
y'_{n+4} &= y'_n + \frac{h}{45}(14\zeta_n + 64\zeta_{n+1} + 24\zeta_{n+2} + 64\zeta_{n+3} + 14\zeta_{n+4}),
\end{aligned} \tag{7}$$

Equation (7) gives the expected family of methods needed to approximate Boundary Value Problems in

Table 1. Comparison of results for the solution of Problem 1

x	ES	CS	Error[8], $p=5$	Error (New Method), $p=5$
0.1	0.147357842331510552	0.147357842560350496	1.100000E-09	2.288399E-10
0.2	0.250152145364362971	0.250152143746982743	3.100000E-09	1.617380E-09
0.3	0.313415043478311325	0.313415040018812160	4.200000E-09	3.459499E-09
0.4	0.341783027465717250	0.341783024082479241	4.210000E-08	3.383238E-09
0.5	0.339543348089778274	0.339543345877885414	8.800000E-08	2.211893E-09
0.6	0.310676924331160645	0.310676921731879408	*1.359000E-07	2.599281E-09
0.7	0.258898185763190914	0.258898182762266000	5.910000E-08	*3.000925E-09
0.8	0.187692247811488679	0.187692245843411027	2.554000E-07	1.968078E-09
0.9	0.100349791960533938	0.100349791600791763	4.533000E-07	3.597422E-10
1.0	0.000000000000000000	0.000000000000000000	4.362000E-07	0.000000E+00

Table 2. Comparison of results for the solution of Problem 2

x	ES	CS	Error[9], $p=5$	Error (New Method), $p=5$
0.125	0.060985349100553900	0.060985349927284837	1.140000E-07	8.267309E-10
0.250	0.138427934741475654	0.138427935540814135	2.220000E-07	7.993385E-10
0.375	0.233175541509714373	0.233175542289879299	*3.200000E-07	7.801650E-10
0.500	0.346110454006479368	0.346110455673130723	3.120000E-07	1.666651E-09
0.625	0.478172624479587739	0.478172626274463905	2.900000E-07	*1.794876E-09
0.750	0.630387283060996859	0.630387283830302552	2.560000E-07	7.693057E-10
0.875	0.803897221213436799	0.803897220949851936	5.910000E-07	2.635849E-10
1.000	1.000000000000000000	1.000000000000000000	N/A	0.000000E+00

Table 3. Comparison of results for the solution of Problem 3

t	ES	CS	Error[9], $p=5$	Error (New Method), $p=5$
0.1	0.059343034025940089	0.059343031623208031	1.130000E-07	2.402732E-09
0.2	0.110134207176555682	0.110134203622495878	2.190000E-07	3.554060E-09
0.3	0.151024408862577146	0.151024403986065658	3.290000E-07	4.876511E-09
0.4	0.180475345562389427	0.180475337884770232	3.740000E-07	7.677619E-09
0.5	0.196734670143683288	0.196734661254339106	4.170000E-07	*8.889344E-09
0.6	0.197807972378616420	0.197807964519196411	*4.680000E-07	7.859420E-09
0.7	0.181427245522797494	0.181427238722625921	4.280000E-07	6.800172E-09
0.8	0.145015397537614513	0.145015389799316420	3.620000E-07	7.738298E-09
0.9	0.085646323767636384	0.085646317937597674	2.620000E-07	5.830039E-09
1.0	0.000000000000000000	0.000000000000000000	N/A	0.000000E+00

the form of (1) above. The method displays uniform order of $p=(5,5,5,5)^t$ with error constant $C_6=\left(\frac{107}{10080},\frac{8}{315},\frac{9}{224},\frac{16}{315}\right)^t$ in the correctors.

3. Numerical Results and Discussion

The following numerical problems are considered for the purpose of showing the accuracy of the new method when compared to previously existing methods.

Problem 1: Consider the special second order boundary value problem from[8],

$$\frac{d^2y}{dx^2}-y=4x-5,\ y(0)=y(1)=0,\ h=0.1$$

with exact solution,

$$y(x)=\frac{7}{7\left(e^2-e^{-2}\right)}\left(e^{2x}-e^{-2x}\right)-\frac{3}{4}x$$

Problem 2: Consider the linear second order boundary value problem from[9]

$$\frac{d^2y}{dx^2} = y + \cos x,\ y(0) = 0,\ y(1) = 1,\ h = 0.125$$

with exact solution,

$$y(x) = \frac{-3\cosh 1 + 3\sinh 1 + \cos 1 + 2}{4\sinh 1} e^{x}$$

$$+ \frac{3\cosh 1 + 3\sinh 1 - \cos 1 - 2}{4\sinh 1} e^{-x} - \frac{\cos x}{2}$$

Problem 3. Consider the general second order boundary value problem from[9],

$$\frac{d^2y}{dt^2} = y' - e^{(t-1)} - 1,\ y(0) = 0,\ y(1) = 0,\ h = 0.1$$

with exact solution,

$$y(t) = t\left(1 - e^{(t-1)}\right)$$

The results and the comparison of errors of these problems are given in the tables as seen in the Appendix.

4. Conclusion

In the tables above, comparison has been made between the four-step method and the methods presented in[8,9]. These methods are all seen to have equal order p = 5 and hence the basis for comparison to show that the new four-step performed better than these previously existing methods despite being of the same order. This is evident in the results presented in the tables above where the numerical results are displayed and the maximum error for these methods marked by the asterisk sign. It is observed that the maximum error for the new four-step block method was better in all the three results presented in Tables 1, 2 and 3. Hence this method is seen to compete favourably with previously existing methods in literature. However, this method has only been adopted to solve Dirichlet boundary conditions, further research will consider solutions of other types of boundary conditions.

5. References

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Appendix

The following notations are used in the tables below:

ES: Exact Solution
CS: Computed Solution