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Novel Conditions on the Non-Normal Cayley Graphs of Valency Six

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Abstract

A Cayley graph $X = Cay(G, S)$ on a group G is said to be normal if the right regular representation $R(G)$ of G is normal in the full automorphism group $Aut(X)$. In this paper, two novel conditions are outlined to identify the non-normal Cayley graphs of valency six. As an application, some non-normal Cayley graphs of valency six on A_6 and A_5 are obtained.

Keywords: Automorphism Groups, Cayley Graph, Normal Cayley Graph

1. Introduction

Let X be a finite simple undirected graph, we use $V(X)$, $E(X)$, $A(X)$ and $Aut(X)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. For every $u,v \in V(X)$, $\{u, v\}$ is the edge incident to u and v in X. A graph is called vertex-transitive if its automorphism group is transitive on the vertex set. A graph is called edge-transitive if its automorphism group is transitive on the edge set. Similarly an arc-transitive graph is a graph whose automorphism group is transitive on the arc set. Throughout this paper the symmetric group of degree n and the alternating group of degree n are denoted by S_n and A_n, respectively.

Let G be a permutation group on a set A and $\alpha \in A$. Denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing the point α. Permutation group G is semiregular on A if $G_\alpha = 1$ for every $\alpha \in A$ and regular if G is transitive and semiregular. Let G be a finite group and let S be a subset of G such that $1 \notin S$ and $S^{-1} = S$. The Cayley graph $X = Cay(G, S)$ on G with respect to S is defined as the graph with a vertex set $V(X) = G$ and an edge set $E(X)=\left\{\{g,h\} \mid g,\ h \in G,\ gh^{-1} \in S\right\}$. A Cayley graph $Cay(G, S)$ is connected if and only if $G = \langle S\rangle$. Let $A = Aut(Cay(G, S))$. It is obvious that $R(G)$ are contained in A. Also is regular on the set $V(X)$. Thus a Cayley graph is vertex transitive. If A_1 denotes the stabilizer of the vertex 1 in A then $Aut(G,S)=\left\{\alpha \in Aut(G) \mid S^\alpha = S\right\}$ is a subgroup of A_1. A Cayley graph $Cay(G, S)$ is said to be normal if $R(G)$ is normal in A.

A lot of study has been done in normality of Cayley graphs. For example, normality of Cayley graphs of order p^2 and $2p$ has been determined by Dobson[4] and Du[5], respectively. Disconnected normal Cayley graphs are highlighted by Wang[9]. Further, Preager[8] has developed a perspective which identifies $Cay(G, S)$ is normal if $N_A(R(G))$ is transitive on edges and $Cay(G, S)$ is a connected cubic Cayley graph on a non-abelian simple group. Also vast majority of normal connected cubic Cayley graphs on non-abelian simple groups are specified by Fang[6].

In 2005, Feng and Xu[7] proved that every connected tetravalent Cayley graph on a regular p-group is normal when $p \neq 2, 5$. One year later, normality of tetravalent Cayley graphs on dihedral groups have been discussed by Wang and Xu[10].

In 2007, normality of the connected Cayley graph of valency 5 on A_5 has been determined by Feng and Zhou[12], although in [11] the normality of the connected Cayley graphs of valency 3 and 4 on A_5 has been proved by Xu and Xu. For more results on the normality of Cayley graphs, we refer the reader to [1–3].

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In this paper, we have presented two main theorems with new conditions in order to ease the identification of non-normal Cayley graphs of valency 6.

2. Preliminaries

First we will give some preliminary results which use in the next.

Let $X = Cay(G, S)$ be a Cayley graph of G with respect to S and $Aut(G,S)=\left\{\alpha \in Aut(G) \mid S^{\alpha}=S\right\}$. Set $A := Aut(X)$ and denote by A_1 the stabilizer of the vertex 1 in A. The following proposition is basic.

Lemma 2.1 [6, Proposition 1.1]
As the above notations:

- **(i)** $Aut(X)$ contains the right regular representation $R(G)$ of G and so X is vertex-transitive.
- **(ii)** X is undirected if and only if $S^{-1} = S$. Hence, all Cayley(di) graphs are vertex-transitive.
- **(iii)** X is connected if and only if $G = \langle S \rangle$.

Lemma 2.2 [6, Proposition 1.3]
We have:

- **(i)** $N_A(R(G)) = R(G)Aut(G, S)$,
- **(ii)** $A = R(G)\ Aut(G, S)$ if and only if $R(G)$ is normal in G.

Lemma 2.3 [11, Proposition 1.5]
The Cayley (di) graph is normal if and only if $A_1 = Aut(G, S)$.

3. Discussion of Main Theorems

Now two sufficient conditions are given on the non-normal Cayley graphs of valency 6 for a finite group.

Theorem 3.1
Let G be a finite group and $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ be a subset of G which $S^{-1} = S$ and s_5, s_6 are involutions. Suppose that S contains at least three involutions and there exists an involution h in $G\backslash S$ such that $s_2 = hs_1$, $s_3 = s_1h$, $s_4 = s_2h$, $s_5 s_6 = h$. $\quad (*)$

Then the Cayley graph $Cay(G, S)$ is not normal.

Proof
By the equations $(*)$ and because $h \notin S$, we have $1 \notin S$. Consider $\sigma=(s_3\ \ s_4)(s_6s_3\ \ s_6s_4)$. Clearly $s_6 s_3 \neq s_3$ and $s_6 s_3 \neq s_4$, because if $s_6 s_3 \neq s_4$ then by the last equation of $(*)$ we have $s_5hs_3 = s_4$ so $s_5s_4 = s_4$ and it implies that $s_5 = 1$, a contradiction. Thus $s_6 s_3 \neq s_4$. Similarly we can see $s_6 s_4 \neq s_3, s_4$. It shows that σ is a permutation on G.

Let $X = Cay(G, S)$ and $A = Aut(X)$. Denote by A_1 the stabilizer of 1 in A. To prove that $X = Cay(G, S)$ is not normal, by Proposition 2.3, it suffices to show that $\sigma \in A_1$ and $\sigma \notin Aut(G, S)$.

By the equations $(*)$, $s_4 = s_2h = hs_1h = s_1^{\,h}$ and $s_3 = s_1 h = hhs_1h = hs_2h = s_2^{\,h}$.

Since s_5s_6 are involutions and by the assumption, S contains at least three involutions, implying that either s_4 and s_1 or s_3 and s_2 are involutions. If s_4 and s_1 are not involutions, then s_3 and s_2 must be involutions. It means that s_3 and s_4 have different orders. If s_4 and s_1 are involutions, then $s_2^{-1} = (hs_1)^{-1} = s_1^{-1}\, h^{-1} = s_1h = s_3$, which means that s_3 and s_2 are not involutions and then s_3 and s_4 have different orders. Thus, $\sigma \notin Aut(G, S)$ because σ permutes s_3 to s_4. Further, $s_6s_3 \neq 1$ and $s_6s_4 \neq 1$ because s_6 is an involution. Hence, σ fixes[1]. So we need only to show that $\sigma \in A = Aut(X)$ and for this, it is enough to show that σ keeps adjacency of edges.

Let $T = \{s_3, s_4, s_6s_3, s_6s_4\}$. For any $\omega \in T$, we have $T = \{\omega, h\omega, s_6\omega, s_6h\omega\}$.

For example if $\omega = s_3$, then

$$\omega = hs_3 = hs_1h = s_2h = s_4$$
$$s_6\omega = s_6s_3$$
$$h\omega = s_6hs_3 = s_6hs_1h = s_6s_4$$

Also if $\omega = s_6s_3$, then

$$h\omega = hs_6s_3 = s_6hs_3 = s_6hs_1h = s_6s_4$$
$$s_6\omega = s_6s_6s_3 = s_3$$
$$s_6h\omega = s_6hs_6s_3 = s_6s_6hs_3 = hs_1h = s_4$$

Similarly, if $\omega = s_4$ or $\omega = s_6s_4$ the same result is obtained. Thus it is assumed that for any $\omega \in T$, $T = \{\omega, h\omega, s_6\omega, s_6h\omega\}$ and $\sigma = (\omega\ h\omega)(s_6\omega\ s_6h\omega)$. Clearly, σ fixes every element in $G\backslash T$.

Now let $\{u, v\} \in E(X)$. We aim to prove that $\{u,v\}^{\sigma} \in E(X)$. Consider two cases:

Case 1. If $\{u, v\} \cap T = \emptyset$, then $\{u, v\} \notin T$ and $\{u, v\}^{\sigma} = \{u, v\} \in E(X)$.

Case 2. If $\{u, v\} \cap T \neq \emptyset$, without loss of generality we can assume $u \in T$, then $T = \{u, hu, s_6u, s_6hu\}$ and $\sigma = (u\ hu)(s_6u s_6hu)$. Thus $u^{\sigma} = hu$, since $E(X)=\left\{(g,sg) \mid g \in G,\ s \in S\right\}$ and $\{u, v\} \in E(X)$, we have $v = s_i\, u$ for some i, where $1 \leq i \leq 6$.

If $v = s_5u$, then $v = s_6hu$ and $\{u, v\}^\sigma = \{hu, s_6u\} \in E(X)$. Similarly if $v = s_6u$, then $\{u,v\}^\sigma = \{hu, s_6hu\} \in E(X)$. Now, suppose that $v = s_ju$ for some j, $1 \le j \le 4$. It is clear that $v = s_j u \ne u$ and $v \ne hu$. Because if $v = hu$, then $s_ju = hu$ for some j, $1 \le j \le 4$. So $s_j = h$ and it is a contradiction. Similarly, $v = s_ju \ne s_6u$ or s_6hu for some j, $1 \le j \le 4$. Therefore $v \notin T$ and $v^\sigma = v$. Now If $j = 1$then $v = s_1 u = s_3hu$ and $\{u,v\}^\sigma = \{hu, s_3 hu\} \in E(X)$. If $j = 2$, then $v = s_2u = hs_1u$ and $\{u,v\}^\sigma = \{hu, hs_1u\} \in E(X)$. Similarly, for $j = 3, 4$ we have $\{u,v\}^\sigma \in E(X)$.

Therefore, both Cases 1, 2 implies that $\sigma \in A$. Thus $\sigma \in A_1$ and σ $Aut(G,S)$, so by Lemma 2.3, $Cay(G,S)$ is not normal.

Theorem 3.2
Let G be a finite group and $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$be a subset of G such that $1 \notin S$, $G = \langle S\rangle$ and $S^{-1} = S$. Suppose that s_1 is an involution, $N = \{1, s_1, s_2, s_3\}$ be a subgroup of G and $H = \langle s_1, s_4, s_5, s_6\rangle$ such that $s_2, s_3 \notin H$. If $|G{:}H| \ge 4$ and $\{s_1s_4, s_1s_5, s_1s_6\} = \{s_4s_1, s_5s_1, s_6s_1\}$, then the Cayley graph $Cay(G,S)$ is not normal.

Proof
Since $|G{:}H| \ge 4$ and $s_1, s_4, s_5, s_6 \in H$, there is a coset Hg such that $s_i \notin Hg$ for each i, $1 \le i \le 6$. It implies that $Hg \ne Hs_i$. Let $X = Cay(G,S)$ and $A = Aut(X)$. Now define a permutation s on G. If $v \in Hg$, then $v^\sigma = s_1v$, and if $v \in G\backslash Hg$, then $v^\sigma = v$. Clearly for each i, $1 \le i \le 6$, $s_i^\sigma = s_i$. Further, $1_\sigma = 1$ because $1 \notin Hg$. If $\sigma \in Aut(G)$, then σ fixes each element of G, because $G = \langle S\rangle$ and $s_i^\sigma = s_i$, and it means $\sigma = 1$, a contradiction. Thus, $\sigma \notin Aut(G,S)$, and it is enough to show that $\sigma \in Aut(X)$.

Let $\{u,v\} \in E(X)$. We claim that $\{u,v\}^\sigma \in E(X)$. For this we consider two cases.

Case 1. If $\{u,v\} \cap Hg = \varnothing$, then $u,v \notin Hg$ and $\{u,v\}^\sigma = \{u,v\}$ so $\sigma \in Aut(X)$.

Case 2. If $\{u,v\} \cap Hg \ne \varnothing$. We may assume that $u \in Hg$, thus $u^\sigma = s_1u$ and since $\{u,v\} \in E(X)$ it is easy to see $v = s_ku$ for some k, $1 \le k \le 6$. If $k = 1$, then $v = s_1u \in Hg$ because $s_1 \in H$ and $u \in Hg$, so we have $\{u,v\}^\sigma = \{s_1u, u\} \in E(X)$. If $k = 2$, then $v = s_2u \notin Hg$ because $s_2 \notin H$. Since $N = \{1, s_1, s_2, s_3\}$ be a group of order 4 and s_1 is an involution, we have $s_2 = s_3s_1$. Therefore, $\{u,v\}^\sigma = \{s_1u, s_2u\} = \{s_1u, s_3s_1 u\} \in E(X)$. Similarly if $k = 3$, then $v = s_3u \notin Hg$ and $\{u,v\}^\sigma = \{s_1u, s_3u\} = \{s_1u, s_2 s_1u\} \in E(X)$. If $k = 4$, then $v = s_4u \in Hg$ because $s_4 \in H$ and $u \in Hg$. So $v^\sigma = s_1s_4u$. By the assumption, we know that $s_1s_4 \in \{s_4s_1, s_5s_1, s_6s_1\}$. Thus there is an s_l such that $s_1 s_4 = s_ls_1$, where $l = 4$ or 5 or 6. So $= \{s_1u, s_1 s_4u\} = \{s_1u, s_ls_1u\} \in E(X)$. Similarly, if $k = 5$, then $v = s_5u \in Hg$ and $\{u,v\}^\sigma = \{s_1u, s_1s_5u\} = \{s_1u, s_r s_1u\} \in E(X)$, where $r = 4$ or 5 or 6. Finally, if $k = 6$, then $v = s_6u \in Hg$ and $\{u,v\}^\sigma = \{s_1 u, s_1s_6u\} = \{s_1u, s_ts_1u\} \in E(X)$, where $t = 4$ or 5 or 6.

It implies that in each case, $\{u,v\}^\sigma \in E(X)$ and so $\sigma \in Aut(X)$. Therefore, $\sigma \in A_1$ but $\sigma \notin Aut(G,S)$ and by Lemma 2.3, $Cay(G,S)$ is not normal.

4. Conclusion

Now we construct an infinite family of non-normal Cayley graphs of valency **6** by using Theorem 3.1 in the following example.

Example 4.1
Let $n(> 2)$ be an even integer and $m > 1$. If $G = \langle a, b, c \mid a^n = b^2 = c^m = 1, b^{-1}ab = a^{-1}, b^{-1}cb = c^{-1}\rangle$, then, the Cayley graph $Cay\left(G, \left\{a^{\frac{n}{2}}, a^{\frac{n}{2}}b, c, c^{-1}, bc, cb\right\}\right)$ is a non-normal Cayley graph of valency **6**

Proof
It is clear that, $a^{\frac{n}{2}}, c, c^{-1} \ne 1$. Further $bc \ne 1$, because if $bc = 1$ then $c = b$ a contradiction. Similarly, $cb \ne 1$. Also $a^{\frac{n}{2}}b \ne 1$, because if $a^{\frac{n}{2}}b = 1$ then $a^{\frac{n}{2}}\left(a^{\frac{n}{2}}b\right) = \left(a^{\frac{n}{2}}a^{\frac{n}{2}}\right)b = a^nb = b \in S$ a contradiction. Thus, $1 \notin S$.

Now, let $h{:} = b \in G \setminus S$ and consider $s_1 = cb$, $s_2 = c^{-1}$, $s_3 = c$, $s_4 = bc$, $s_5 = a^{\frac{n}{2}}$ and $s_6 = a^{\frac{n}{2}}b$. It is easy to see that h is an involution, S has at least three involutions and $s_5s_6 = h$, $s_2 = s_1h$, $s_4 = s_2h$. Thus the conditions of Theorem 3.1 are hold and $Cay(G,S)$ is not normal.

In following examples some non-normal Cayley graphs of valency **6** on A_6 and A_5 are determined.

Example 4.2
Let $W_1 = \{(1\ 2)\ (4\ 5), (5\ 6\ 4), (4\ 6\ 5), (6\ 4)(1\ 2), (5\ 6)(3\ 4), (1\ 2)(3\ 4)\}$. Then the $Cay(A_6, W_1)$ is not normal.

Proof
Consider $h = (1\ 2)(5\ 6)$. It is clear that $h \in A_6\backslash W_1$ and h is an involution. Now suppose that $s_1 = (1\ 2)(4\ 5)$, $s_2 = (5\ 6\ 4)$, $s_3 = (4\ 6\ 5)$, $s_4 = (6\ 4)(1\ 2)$, $s_5 = (5\ 6)(3\ 4)$ and $s_6 = (1\ 2)\ (3\ 4)$. It is easy to see the conditions of Theorem 3.1 are hold and the Cayley graph $Cay(A_6, W_1)$ is not normal.

Example 4.3
Let W_2 = {(1 3) (2 6), (5 2 6), (2 5 6), (5 2)(1 3), (5 6)(2 4), (1 3) (2 4)}. Then the $Cay(A_6, W_2)$ is not normal.

Proof
Consider h = (1 3)(5 6). It is clear that $h \in A_6 \backslash W_2$ and h is an involution. Let s_1 = (1 3)(2 6), s_2 = (5 2 6), s_3 = (2 5 6), s_4 = (5 2)(1 3), s_5 = (5 6)(2 4) and s_6 = (1 3)(2 4). It is easy to see the conditions of Theorem 3.1 are hold and the Cayley graph $Cay(A_6, W_2)$ is not normal.

Example 4.4
Let W_3 = {(1 4)(2 6),(3 2 6),(2 3 6), (3 2)(1 4), (3 6)(2 5), (1 4)(2 5)}. Then the $Cay(A_6, W_3)$ is not normal.

Proof
Similarly, by consider h = (1 4)(3 6), the $Cay(A_6, W_3)$ is not normal.

Example 4.5
Let W_4 = {(1 5)(3 4), (2 3 4), (3 2 4), (2 3)(1 5), (2 4)(3 6), (1 5)(3 6)}. Then the $Cay(A_6, W_4)$ is not normal.

Proof
Similarly, by consider h = (1 5)(2 4), the Cay(A_6, W_4) is not normal.

Example 4.6
Let W_5 = {(1 6)(3 5), (2 5 3), (3 5 2), (2 5)(1 6), (2 3)(4 5), (1 6)(4 5)}. Then the $Cay(A_6, W_5)$ is not normal.

Proof
Similarly, by consider h = (1 6)(2 3), the $Cay(A_6, W_5)$ is not normal.

Example 4.7
Let U_1 = {(1 2)(4 5), (3 5 4), (4 5 3), (3 5)(1 2), (1 4)(2 3), (1 3)(2 4)}. Then the $Cay(A_5, U_1)$ is not normal.

Proof
Consider h = (1 2)(3 4). It is clear that $h \in A_5 \backslash U_1$ and h is an involution. Let s_1 = (1 2)(4 5), s_2 = (3 5 4), s_3 = (4 5 3), s_4 = (3 5)(1 2), s_5 = (1 4)(2 3) and s_6 = (1 3)(2 4). It is easy to see that and $s_2 = hs_1$, $s_3 = s_1h$, $s_4 = s_2h$ and $s_6 s_5 = h$. So by the Theorem 3.1, the Cay(A_5,U_1) is not normal.

Example 4.8
Let U_2 = {(1 5)(2 4), (2 3 4),(2 4 3), (3 4)(1 5), (1 3)(2 5), (1 2)(5 3)}. Then the Cay(A_5,U_2) is not normal.

Proof
Consider h =(1 5)(2 3). It is clear that $h \in A_5 \backslash U_2$ and h is an involution. Similarly we have $s_2 = hs_1$, $s_3 = s_1h$, $s_4 = s_2h$ and $s_6 s_5 = h$. So by the Theorem 3.1, the $Cay(A_5,U_2)$ is not normal.

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