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Some Properties of a New Class of Univalent Functions Involving a New Generalized Differential Operator with Negative Coefficients

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Abstract

The primary motivation behind this paper is to present new generalized differential operator $A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)$ defined through $\mathrm{U}=\{z\in\mathbb{C}:|z|<1\}$ we present a new contribution is subclass of analytic functions $G_m(\gamma,\sigma,\chi)$. Additionally, we talk about some properties for univalent functions with many results for the subclass of analytic functions $G_m(\gamma,\sigma,\chi)$. Furthermore, with given solve technical to application involving fractional calculus for univalent functions.

Keywords: Analytic Functions, Differential Operator, Close-to-convex Functions, Fractional Calculus, Starlike Functions

1. Introduction and Preliminaries

Now, we will go to present preface for univalent functions, with important contributions to get some important properties in this is article.

Let A denote the class of univalent function $f(z)$:

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n,\quad (a_n\geq 0) \tag{1}$$

which areanalyticfunction in the unit disc $\mathrm{U}=\{z\in\mathbb{C}:|z|<1\}$. For a univalent function f in A, we introduce the following differential operator

$$A^0_{\mu,\lambda,\delta}(\alpha,\beta)f(z)=f(z),$$

$$A^1_{\mu,\lambda,\delta}(\alpha,\beta)f(z)=\left(1-\frac{\beta(\lambda-\alpha)}{\mu+\lambda}\right)f(z)$$

$$+\left(\frac{\beta(\lambda-\alpha)}{\mu+\lambda}\right)zf'(z)+\frac{\delta}{\mu+\lambda}z^2f''(z),$$

and for $m=1,2,3,\ldots$

$$A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)=A^m_{\mu,\lambda,\delta}(\alpha,\beta)\left(A^{m-1}_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right). \tag{2}$$

If f is given by (1), then from (2) we get

$$A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)=$$

$$z+\sum_{n=2}^{\infty}\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^m a_nz^n \tag{3}$$

for $f\in$ A, $.\alpha,\delta\geq 0$, $\beta,\lambda,\mu>0$, $\alpha\neq\lambda$ and $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

This generalizesnumerous operators as follows.

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(i) $A^m_{\mu,\lambda,0}(0,1)f(z)=z+\sum_{n=2}^{\infty}\left[\frac{\mu+\lambda n}{\mu+\lambda}\right]^m a_n z^n$, the operator presented and studied by Swamy[12],

(ii) $A^m_{1-\lambda,\lambda,0}(\alpha,\beta)f(z)=z+\sum_{n=2}^{\infty}\left[1+(n-1)(\lambda-\alpha)\beta\right]^m a_n z^n$ the operator presented and studied by Darus and Ibrahim[4],

(iii) $A^m_{1-\lambda,\lambda,0}(0,1)f(z)=z+\sum_{n=2}^{\infty}\left[1+(n-1)\lambda\right]^m a_n z^n$ the operator presented and studied by Al-Oboudi[5],

(iv) $A^m_{0,1,0}(0,1)f(z)=z+\sum_{n=2}^{\infty}n^m a_n z^n$, the operator presented and studied by Salagean[2],

Let $G_m(\gamma,\sigma,\chi)$ denote andpresent a new contribution is subclass of A comprising ofunivalent functions f which satisfy

$$\left|\frac{\gamma\left[\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'-\frac{A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{z}\right]}{\sigma\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'-(1-\gamma)\frac{A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{z}}\right|<\chi, \tag{4}$$

where $A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)$ is given by (3), $0\le\gamma<1, 0\le\delta<1, 0<\chi<1$ and for all $z\in U$.

We know that $G_0(\gamma,\sigma,\chi)$ was presented by Atshan and Ghawi[6]. We use techniques similar to these used Amourah et al.[1],Darus and Faisal[3], Al-Hawary et al.[7] and [13–15].

2. Some Results for the Class $G_m(\gamma,\sigma,\chi)$

In this part inarticle, we find the first important result coefficientine quality

Theorem 2.1 A univalent function $f\in A$ is in the class $G_m(\gamma,\sigma,\chi)$ if and only if

$$\sum_{n=2}^{\infty}\left[\gamma(n-1)+\chi(n\sigma+1-\gamma)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^m a_n\le\chi\left(\sigma+(1-\gamma)\right). \tag{5}$$

The result (5) is sharp.

Proof. Suppose that the result (5) hold true and $|z|=1$. Now, we follow technique steps

$$\left|\gamma\left[\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'-\frac{A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{z}\right]\right|-\chi\left|\sigma\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'-(1-\gamma)\frac{A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{z}\right|$$

$$=\left|-\gamma\sum_{n=2}^{\infty}(n-1)\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^m a_n z^{n-1}\right|-\chi\left|\sigma+(1-\gamma)-\sum_{n=2}^{\infty}(n\sigma+1-\gamma)\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^m a_n z^{n-1}\right|$$

$$\le\sum_{n=2}^{\infty}\left[\gamma(n-1)+\chi(n\sigma+1-\gamma)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^m a_n-\chi\left(\sigma+(1-\gamma)\right)\le 0.$$

Hence, by maximum modules principle, $f\in G_m(\gamma,\sigma,\chi)$.Now, suppose that $f\in G_m(\gamma,\sigma,\chi)$

Then we have from (3) that

$$\left|\gamma\left[\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'-\frac{A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{z}\right]\right|<\chi\left|\sigma\left(A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)\right)'-(1-\gamma)\frac{A^m_{\mu,\lambda,\delta}(\alpha,\beta)f(z)}{z}\right|,$$

we get

$$=\left|-\gamma\sum_{n=2}^{\infty}(n-1)\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^m a_n z^{n-1}\right|<\chi\left|\sigma+(1-\gamma)-\sum_{n=2}^{\infty}(n\sigma+1-\gamma)\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^m a_n z^{n-1}\right|.$$

Thus

$$\sum_{n=2}^{\infty}\left[\gamma(n-1)+\chi\left(n\sigma+1-\gamma\right)\right]$$

$$\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^{m}a_n\le\chi\left(\sigma+(1-\gamma)\right).$$

This implies that $f\in G_m\left(\gamma,\sigma,\chi\right)$.The result for coefficientinequality (5) is sharp for the univalent function

$$f(z)=z-\frac{\chi\left(\sigma+(1-\gamma)\right)}{\left[\gamma(n-1)+\chi\left(n\sigma+1-\gamma\right)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^{m}}z^n(n\ge 2).$$

Theorem 2.2 If the function f defined by (1) is in the class $G_m\left(\gamma,\sigma,\chi\right)$, then for $|z|<1$, we have

$$|z|-\frac{\chi\left(\sigma+(1-\gamma)\right)}{\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^{m}}|z|^2\le|f(z)|\le$$
$$|z|+\frac{\chi\left(\sigma+(1-\gamma)\right)}{\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^{m}}|z|^2 \quad (6)$$

and

$$1-\frac{2\chi\left(\sigma+(1-\gamma)\right)}{\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^{m}}|z|\le|f'(z)|\le$$
$$1+\frac{\chi\left(\sigma+(1-\gamma)\right)}{\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^{m}}|z| \quad (7)$$

Proof. It is easy to see that, for $G_m\left(\gamma,\sigma,\chi\right)$,

$$\sum_{n=2}^{\infty}a_n\le\frac{\chi\left(\sigma+(1-\gamma)\right)}{\left[\gamma(n-1)+\chi\left(n\sigma+1-\gamma\right)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^{m}}$$

And

$$\sum_{n=2}^{\infty}na_n\le\frac{2\chi\left(\sigma+(1-\gamma)\right)}{\left[\gamma(n-1)+\chi\left(n\sigma+1-\gamma\right)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^{m}},$$

$$\sum_{n=2}^{\infty}\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^{m}a_n$$
$$\le\sum_{n=2}^{\infty}\left[\gamma(n-1)+\chi\left(n\sigma+1-\gamma\right)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^{m}a_n,(n\ge 2)$$

And

$$\sum_{n=2}^{\infty}\frac{n}{2}\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^{m}a_n$$
$$\le\sum_{n=2}^{\infty}\left[\gamma(n-1)+\chi\left(n\sigma+1-\gamma\right)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^{m}a_n,(n\ge 2)$$

we have,

$$|f(z)|\le|z|+|z|^2\sum_{n=2}^{\infty}a_n$$
$$\le|z|+\frac{\chi\left(\sigma+(1-\gamma)\right)}{\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^{m}}|z|^2$$

$$|f(z)|\ge|z|-|z|^2\sum_{n=2}^{\infty}a_n\ge|z|-\frac{\chi\left(\sigma+(1-\gamma)\right)}{\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^{m}}|z|^2$$

$$\left|f'(z)\right| \le 1+|z|\sum_{n=2}^{\infty} na_n \ge |z|$$

$$+\frac{2\chi\left(\sigma+(1-\gamma)\right)}{\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^m}|z|$$

$$\left|f'(z)\right| \ge 1-|z|\sum_{n=2}^{\infty} na_n \ge |z|$$

$$-\frac{2\chi\left(\sigma+(1-\gamma)\right)}{\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^m}|z|$$

3. Close-to-convexity, Star likeness and Convexity

In this part in article for a univalent function $f(z)$,we find the second important result for $f(z)\in A$ is said to be close-to-convex of order η if it satisfies

$$\operatorname{Re}\left\{f'(z)\right\}>\eta \tag{8}$$

for some $\eta\,(0\le\eta\le 1)$) and for all $z\in U$. Also a univalent function $f(z)\in A$ is said to be starlike of order η if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\eta \tag{9}$$

for some ($\eta\,(0\le\eta\le 1)$) and for all $z\in U$. Additional, a univalent function $f(z)\in A$ is said to be convex of order η , if and only if $zf'(z)$ is starlike of order η , that is if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\eta \tag{10}$$

for some $\eta\,(0\le\eta\le 1)$ and for all $z\in U$.

Theorem 3.1 If $f(z)\in G_m(\gamma,\sigma,\chi)$, then a univalent function $f(z)$ is close-to-convex of order η in $|z|<h_1(\sigma,\chi,\gamma,\eta)\in G_m(\sigma,\chi,\gamma,\eta)$, where,

$$h_1(\sigma,\chi,\gamma,\eta)=\inf_n \left\{\frac{(1-\eta)\left[\gamma(n-1)+\chi(n\sigma+1-\gamma)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^m}{n\chi\left(\sigma+(1-\gamma)\right)}\right\}^{\frac{1}{n-1}}.$$

Proof. It is follow technique steps adequate to demonstrate that

$$\left|f'(z)-1\right|<\sum_{n=2}^{\infty} na_n|z|^{n-1}\le 1-\eta \tag{11}$$

And

$$\sum_{n=2}^{\infty}\left[\gamma(n-1)+\chi(n\sigma+1-\gamma)\right]$$

$$\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^m a_n\le\chi\left(\sigma+(1-\gamma)\right).$$

observe that (11) is true if

$$\frac{n|z|^{n-1}}{1-\eta}\le\frac{\left[\gamma(n-1)+\chi(n\sigma+1-\gamma)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^m}{\chi\left(\sigma+(1-\gamma)\right)}. \tag{12}$$

Solving (12) for $|z|$, we obtain

$$|z|\le\left\{\frac{(1-\eta)\left[\gamma(n-1)+\chi(n\sigma+1-\gamma)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^m}{n\chi\left(\sigma+(1-\gamma)\right)}\right\}^{\frac{1}{n-1}}$$

(n = 2, 3, …).

Theorem 3.2 If $f(z)\in G_m(\gamma,\sigma,\chi)$, then a univalent function $f(z)$ is starlike of order η in $|z|<h_2(\sigma,\chi,\gamma,\eta)\in G_m(\sigma,\chi,\gamma,\eta)$, where

$$h_2(\sigma,\chi,\gamma,\eta)=\inf_n \left\{\frac{(1-\eta)\left[\gamma(n-1)+\chi(n\sigma+1-\gamma)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^m}{(n-\eta)\chi\left(\sigma+(1-\gamma)\right)}\right\}^{\frac{1}{n-1}}.$$

Proof. We should get that $\left|\frac{zf'(z)}{f(z)}-1\right|<1-\eta$

For $|z|<h_2(\sigma,\chi,\gamma,\eta)$. Since

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le\frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}}$$

If

$$\frac{(n-\eta)|z|^{n-1}}{1-\eta}\le\frac{\left[\gamma(n-1)+\chi(n\sigma+1-\gamma)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^m}{\chi(\sigma+(1-\gamma))},$$

$f(z)$ is star like of order η .

Corollary 3.3 If $f(z)\in G_m(\gamma,\sigma,\chi)$, then a univalent function $f(z)$ isconvexoforder η in $|z|<h_3(\sigma,\chi,\gamma,\eta)$, where

$$h_3(\sigma,\chi,\gamma,\eta)=\inf_n$$
$$\left\{\frac{(1-\eta)\left[\gamma(n-1)+\chi(n\sigma+1-\gamma)\right]\left[1+\frac{(n-1)\left[(\lambda-\alpha)\beta+n\delta\right]}{\mu+\lambda}\right]^m}{n(n-\eta)\chi(\sigma+(1-\gamma))}\right\}^{\frac{1}{n-1}}$$

4. An Application in the Fractional Calculus

In this part in article for a univalent function $f(z)$, we find the third important result for $f(z)\in A$ is,Owa[8]gave the accompanying definitions for the fractional calculus. For else definitions[9–11].

Definition 4.1 The fractional integral of order υ is defined by

$$D_z^{-\upsilon}f(z)=\frac{1}{\Gamma(\upsilon)}\int_0^z\frac{f(t)}{(z-t)^{1-\upsilon}}dt$$

where, $\upsilon>0$. $f(z)$ is analytic function in a simply connected region of the $z-$plane containing the origin and multiplicity of $(z-t)^{\upsilon-1}$ is remove by requiring $Log(z-t)$ to be real when $(z-t)>0$.

Definition 4.2 The fractional derivative of order υ is defined by

$$D_z^{\upsilon}f(z)=\frac{1}{\Gamma(1-\upsilon)}\frac{d}{dz}\int_0^z\frac{f(t)}{(z-t)^{\upsilon}}dt,$$

where, $0\le\upsilon<1$. $f(z)$ is analytic function in a simply connected region of the $z-$plane containing the origin and multiplicity of $(z-t)^{-\upsilon}$ is remove as in Definition 4.1.

Definition 4.3 Under the conditions of Definition 4.2, the fractional derivative of order $n+\upsilon$ is defined by $D_z^{n+\upsilon}f(z)=\frac{d^n}{dz^n}D_z^{\upsilon}f(z)$, where $0\le\upsilon<1$ and n = 0, 1, 2, … .

Theorem 4.4 Let the function $f(z)$ be in the class $G_m(\gamma,\sigma,\chi)$. Then

$$\left|D_z^{-\upsilon}f(z)\right|\le\frac{1}{\Gamma(2+\upsilon)}|z|^{1+\upsilon}$$
$$\left[1+\frac{2\chi(\sigma+(1-\gamma))}{(2+\upsilon)\left[\gamma+\chi(2\sigma+1-\gamma)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^m}|z|\right]\quad(13)$$

And

$$\left|D_z^{-\upsilon}f(z)\right|\le\frac{1}{\Gamma(2+\upsilon)}|z|^{1+\upsilon}$$
$$\left[1-\frac{2\chi(\sigma+(1-\gamma))}{(2+\upsilon)\left[\gamma+\chi(2\sigma+1-\gamma)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^m}|z|\right]\quad(14)$$

$$\left(\chi(\sigma+(1-\gamma))\le\frac{(2+\upsilon)\left[\gamma+\chi(2\sigma+1-\gamma)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^m}{2\Gamma(2+\upsilon)}\right).$$

The last equalities in (13) and (14) are accomplished for the function

$$f(z)=z-\frac{\chi(\sigma+(1-\gamma))}{\left[\gamma+\chi(2\sigma+1-\gamma)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^m}z^2.$$

Proof. By use the first important result coefficient ©in equality in Theorem 2.1

$$\sum_{n=2}^{\infty} a_n \le \frac{\chi\left(\sigma+(1-\gamma)\right)}{\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^m}. \tag{15}$$

From Definition 4.1 we get

$$D_z^{-\upsilon} f(z)=\frac{1}{\Gamma(2+\upsilon)}z^{1+\upsilon}-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+\upsilon+1)}a_n z^{n+\upsilon}$$

And

$$\Gamma(2+\upsilon)z^{-\upsilon}D_z^{-\upsilon}f(z)$$

$$=z-\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(\upsilon+2)}{\Gamma(n+\upsilon+1)}a_n z^n=z-\sum_{n=2}^{\infty}\Psi(n)a_n z^n \tag{16}$$

where

$$\Psi(n)=\frac{\Gamma(n+1)\Gamma(\upsilon+2)}{\Gamma(n+\upsilon+1)}.$$

We get that $\Psi(n)$ is a decreasing for a univalent function of n and $0<\Psi(n)\le\Psi(2)=\frac{2}{2+\upsilon}$. Using(15) and (16) we get

$$\left|\Gamma(2+\upsilon)z^{-\upsilon}D_z^{-\upsilon}f(z)\right|\le|z|+\Psi(2)|z|^2\sum_{n=2}^{\infty}a_n$$

$$\le|z|+\frac{2\chi\left(\sigma+(1-\gamma)\right)}{(2+\upsilon)\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^m}|z|^2,$$

which gives (13), we likewise have

$$\left|\Gamma(2+\upsilon)z^{-\upsilon}D_z^{-\upsilon}f(z)\right|\ge|z|-\Psi(2)|z|^2\sum_{n=2}^{\infty}a_n$$

$$\ge|z|-\frac{2\chi\left(\sigma+(1-\gamma)\right)}{(2+\upsilon)\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^m}|z|^2$$

which gives (14).

Theorem 4.5 Let the function $f(z)$ be in the class $G_m(\gamma,\sigma,\chi)$. Then,

$$\left|D_z^{\upsilon}f(z)\right|\le\frac{1}{\Gamma(2-\upsilon)}|z|^{1-\upsilon}\left[1+\frac{2\chi\left(\sigma+(1-\gamma)\right)}{(2-\upsilon)\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^m}|z|\right] \tag{17}$$

And

$$\left|D_z^{\upsilon}f(z)\right|\ge\frac{1}{\Gamma(2-\upsilon)}|z|^{1-\upsilon}\left[1-\frac{2\chi\left(\sigma+(1-\gamma)\right)}{(2-\upsilon)\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^m}|z|\right] \tag{18}$$

$$\left(\chi\left(\sigma+(1-\gamma)\right)\le\frac{(2-\upsilon)\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^m}{2\Gamma(2-\upsilon)}\right).$$

The equalities in (17) and (18) are accomplished for a univalent function

$$f(z)=z-\frac{\chi\left(\sigma+(1-\gamma)\right)}{\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^m}z^2.$$

Proof. Using Theorem 2.1 we have

$$\sum_{n=2}^{\infty} na_n \le \frac{2\chi\left(\sigma+(1-\gamma)\right)}{\left[\gamma+\chi\left(2\sigma+1-\gamma\right)\right]\left[1+\frac{\left[(\lambda-\alpha)\beta+2\delta\right]}{\mu+\lambda}\right]^m}. \tag{19}$$

From Definition 4.2 we get

$$D_z^{\upsilon} f(z) = \frac{1}{\Gamma(2-\upsilon)} z^{1-\upsilon} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-\upsilon+1)} a_n z^{n-\upsilon}$$

And

$$\Gamma(2-\upsilon) z^{\upsilon} D_z^{\upsilon} f(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\upsilon)}{\Gamma(n-\upsilon+1)} a_n z^n$$

$$= z - \sum_{n=2}^{\infty} \frac{\Gamma(n)\Gamma(2-\upsilon)}{\Gamma(n-\upsilon+1)} n a_n z^n = z - \sum_{n=2}^{\infty} n\Phi(n) a_n z^n \tag{20}$$

since $\Phi(n) = \frac{\Gamma(n)\Gamma(2-\upsilon)}{\Gamma(n-\upsilon+1)}$

We know that $\Phi(n)$ is a decreasing for a univalent function of n and $0 < \Phi(n) \le \Phi(2) = \frac{2}{2-\upsilon}$

Using(19) and (20) we have

$$\left|\Gamma(2-\upsilon) z^{\upsilon} D_z^{\upsilon} f(z)\right| \le |z| + \Phi(2)|z|^2 \sum_{n=2}^{\infty} n a_n \le$$

$$|z| + \frac{2\chi\left(\sigma + (1-\gamma)\right)}{(2-\upsilon)\left[\gamma + \chi\left(2\sigma + 1 - \gamma\right)\right]\left[1 + \frac{\left[(\lambda-\alpha)\beta + 2\delta\right]}{\mu+\lambda}\right]^m} |z|^2,$$

which gives (17), we also have

$$\left|\Gamma(2-\upsilon) z^{\upsilon} D_z^{\upsilon} f(z)\right| \le |z| - \Phi(2)|z|^2 \sum_{n=2}^{\infty} n a_n$$

$$\le |z| - \frac{2\chi\left(\sigma + (1-\gamma)\right)}{(2-\upsilon)\left[\gamma + \chi\left(2\sigma + 1 - \gamma\right)\right]\left[1 + \frac{\left[(\lambda-\alpha)\beta + 2\delta\right]}{\mu+\lambda}\right]^m} |z|^2,$$

which gives (18).

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