Error Analysis of Heat Conduction Partial Differential Equations using Galerkin's Finite Element Method

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Abstract

The present work explores an error analysis of Galerkin finite element method (GFEM) for computing steady heat conduction in order to show its convergence and accuracy. The steady state heat distribution in a planar region is modeled by two-dimensional Laplace partial differential equations. A simple three-node triangular finite element model is used and its derivation to form elemental stiffness matrix for unstructured and structured grid meshes is presented. The error analysis is performed by comparison with analytical solution where the difference with the analytical result is represented in the form of three vector norms. The error analysis for the present GFEM for structured grid mesh is tested on heat conduction problem of a rectangular domain with asymmetric and mixed natural-essential boundary conditions. The accuracy and convergence of the numerical solution is demonstrated by increasing the number of elements or decreasing the size of each element covering the domain. It is found that the numerical result converge to the exact solution with the convergence rates of almost $O(h^2)$ in the Euclidean L^2 norm, $O(h^2)$ in the discrete perpetuity L^{∞} norm and $O(h^1)$ in the H^1 norm.

Keywords: Error Analysis, Finite Element Method, Galerkin's Weight Residual Approach, Heat Conduction, Laplace Equation, Partial Differential Equation

1. Introduction

Finite Element Method (FEM) is one of the powerful numerical approaches to solve Partial Differential Equations (PDE). FEM is commonly used in multidirectional fields to solve partial differential equation problems occurring in solid mechanics, biomechanics, fluid mechanics, electromagnetic, thermodynamics etc.¹⁻⁸ The Galerkin's Finite Element Method (GFEM) is one of the weight residual methods. In this weight residual method, an approximating function called trial or basis function⁹⁻¹¹ satisfying elemental boundary conditions is substituted into the given differential equation to give the residual function. The residual is then weighted and the integral of the product, taken over the domain, is set to zero. In the GFEM method, the weighted function is constructed based on the first derivative of the trial function with respect to the nodal variables¹². For this

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reason, GFEM is perhaps the most appropriate solution of PDE using weak formulation¹². Therefore, in GFEM, the governing PDE is first developed in the form of the weak formulation. It is also called a variational formulation of the problem or the method of weight residuals. This follows by the integration of the residual over the whole domain and, if necessary, the Green's integration on the boundary. The integration on the domain is performed by discretizing geometrically into as many finite elements as required. Each element has their nodal coordinates and nodal variables. For each of this element, the Galerkin's approximation of the given PDE is selected by taking into account the nodal coordinates and variables. After that, the integration of each element is performed resulting in the element matrix formulation as well as the boundary condition vector matrix. Finally, the system of liner equations is solved to examine the quality of the approximation solutions. The current work attempts to implement the GFEM procedure above to solve the Laplace partial differential equations (time independent with no heat source) in a rectangular domain where the exact solution is available. Considering essential and natural boundary conditions for the solution domain, the non-linear partial differential equations are solved. A simplified stiffness matrix that can be used for a homogenous rectangular domain problem that allow for a structured grid mesh generation with uniform distribution of element sizes is presented where its scheme can reduce significantly the CPU time.

2. Mathematical Formulation

We consider a steady state heat conduction/flow problem with no heat source, in a homogeneous domain that leads to Laplace's equation which can be combined with inhomogeneous Dirichlet or Neumann conditions as shown in Figure 1^{8-11} .



Figure 1. Physical domain of Ω bounded by Γ .

The leading partial differential equation for a steady state heat conduction can be expressed as

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) = 0 \quad in \quad \Omega \qquad (1)$$
$$0 < x < a \quad and \quad 0 < y < b$$

with the boundary conditions:

$$u = u_b$$
 on Γ_e
 $\frac{\partial u}{\partial n} = f_b$ on Γ_n (2)

where, u_b and f_b are the prescribed essential / Dirichlet and natural / Neumann boundary conditions, respectively. The strong formulation of the weighted residual of the PDE and its boundary of Eqs. (1) and (2) can be written as

$$I = \int_{\Omega} w \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) d\Omega - \bigwedge_{\Gamma} w \frac{\partial u}{\partial n} d\Gamma \quad (3)$$

where, *w* is the weighted function formulated using Galerkin approach. The weak formulation of Eq. (3) can be performed by integration by part to give

$$I = -\int_{\Omega} w \left(\frac{\partial w}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial u}{\partial y} \right) d\Omega + \prod_{\Gamma} w \frac{\partial u}{\partial n} d\Gamma$$
(4)

In the GFEM, the domain Ω is discretized into finite elements and the integration is performed per element to form the so called stiffness matrix.

3. Basis Function and Stiffness Matrix

One of the simplest two-dimensional elements is the three-node triangular linear elements. The basic element suitable for unstructured grid mesh is shown in Figure 2(a). Assume that each triangular element has three nodes (x_1, y_1) , (x_2, y_2) and (x_3, y_3) and their three nodal variables u_1 , u_2 and u_3 at the vertices of the triangle. The value of the variable u at arbitrary location (x, y) in the elemental triangular domain region is approximated by the interpolation/basis function as follows¹²:

$$u = H_1(x, y)u_1 + H_2(x, y)u_2 + H_3(x, y)u_3$$
 (5)

where, $H_i(x, y)$ is the shape function for linear triangular element which can be derived as functions of the three triangular geometry coordinates as follows:

$$H_1 = \frac{1}{2A} \Big[(x_2 y_3 - x_3 y_2) + (y_2 - y_3) x + (x_3 - x_2) y \Big]$$
⁽⁶⁾

$$H_{2} = \frac{1}{2A} \Big[(x_{2}y_{1} - x_{1}y_{3}) + (y_{3} - y_{1})x + (x_{1} - x_{3})y \Big]$$
(7)

$$H_{3} = \frac{1}{2A} \Big[(x_{1}y_{2} - x_{2}y_{1}) + (y_{1} - y_{2})x + (x_{2} - x_{1})y \Big]$$
(8)

and A is the triangular geometric area which can be evaluated as

$$A = \frac{1}{2} \Big[(y_1 + y_2)(x_1 - x_2) + (y_2 + y_3)(x_2 - x_3) + (y_3 + y_1)(x_3 - x_1) \Big]$$
(9)

Equations (5) – (12) show that the trial function u represents all the three nodal variables at the vertices

through the shape function H. The Galerkin approach adopts the assumption that the weight function w in Eq. (4) is the derivative of the trial function with respect to the nodal variables as

$$w = \frac{\partial u}{\partial u_i}$$
(10)

Figure 2. Linear triangular elements (a) For unstructured grid mesh (b) for structured grid mesh.

such that, by considering the first integral of the weak formulation given in Eq. (4), the elemental stiffness matrix can be derived as

$$K^{e} = \int_{\Omega^{e}} \left(\frac{\partial w}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial u}{\partial y} \right) \partial \Omega$$
(11)
$$\left[\left(\frac{\partial H_{1}}{\partial x} \right) \qquad \left(\frac{\partial H_{1}}{\partial x} \right) \right]$$

$$K^{e} = \int_{\Omega^{e}} \left[\frac{\overline{\partial x}}{\partial A} \\ \frac{\partial H_{2}}{\partial x} \\ \frac{\partial H_{3}}{\partial y} \\ \frac{\partial H_{3$$

where, Ω^{e} is the element domain. It can be shown that the elemental stiffness matrix is a 3 by 3 symmetric matrix as follow

$$\begin{bmatrix} K^e \end{bmatrix} = \frac{1}{4A} \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{32} & k_{32} & k_{33} \end{bmatrix}$$
(12)

where the diagonal elements of the stiffness matrix are

$$k_{11} = (x_3 - x_2)^2 + (y_3 - y_2)^2$$

$$k_{22} = (x_1 - x_3)^2 + (y_3 - y_1)^2$$

$$k_{33} = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
(13)

and the off-diagonal elements of the stiffness matrix are

$$k_{12} = (x_3 - x_2)(x_1 - x_3) + (y_3 - y_2)(y_1 - y_3)$$

$$k_{13} = (x_3 - x_2)(x_2 - x_1) + (y_3 - y_2)(y_2 - y_1)$$

$$k_{23} = (x_1 - x_3)(x_2 - x_1) + (y_1 - y_3)(y_2 - y_1)$$
(14)

with the symmetric properties of the off-diagonal element as

$$k_{21} = k_{12}, \quad k_{31} = k_{13}, \quad k_{23} = k_{32}$$
 (15)

The element stiffness matrix given in Equations (13) and (14) is applicable for unstructured grid mesh with arbitrary geometry coordinates of the three vertices as shown in Fig. 2a. For a structured grid mesh, the triangular has one right angle such as shown in Fig. 2b. For this type of mesh, the stiffness matrix can be further simplified by taking into account the advantage that the adjacent sides of the right angle are parallel to the x and y coordinate axes. For example, for the triangular element shown in Fig. 2b, the diagonal element of the stiffness matrix can be simplified as

$$k_{11} = d^{2}$$

$$k_{22} = b^{2} + d^{2}$$

$$k_{33} = b^{2}$$
(16)

and the off-diagonal matrix elements can be simplified to

$$k_{12} = -d^{2}$$

$$k_{13} = 0$$

$$k_{23} = -b^{2}$$
(17)

where the triangular element area is simply $A = \frac{1}{2}b d$

For a homogenous rectangular domain where a structured grid mesh can be performed with a uniform distribution of the length b and height d of each element, the element stiffness matrix given in Equations (16) and (17) is actually the same for each element having the same orientation. Therefore, the stiffness matrix needs to be calculated one time only for each type of orientation. This scheme will significantly reduce CPU time especially if huge number of finite elements is used since no need to construct the stiffness matrix for each element.

4. Performance Evaluation and Result Discussion

This section presents a number of numerical approximation examples to show the features of the GFEM based on triangular element and linear interpolation/basis function for Laplace partial differential equations. The steady state heat conduction problem of rectangular domain shown in Figure 3 is used. The length and width of domain are 10 and 5 respectively. The Dirichlet and Neuman boundary conditions imposed are as follows:

$$u(0, y) = 0$$

$$u(x, 0) = 0$$

$$u(x, 10) = 100 \cdot \sin(0.1\pi x)$$

$$\frac{\partial u}{\partial x}(5, y) = 0$$



Figure 3. Steady state heat conduction problem of rectangular domain.

This problem has been presented in ¹² by using triangular linear element. In the present work, the problem is solved using the simplified stiffness matrix of the triangular element with the modified scheme of the stiffness calculation. The whole rectangular Cartesian mesh are refined or the elements sizes will be decreased as large error is usually found within the elements¹³. The convergence of the Galerkin method solution and exact solution are shown in Figure 4. Note that the exact solution to the problem within the rectangular domain is as follow¹².

$$u = 100 \sin(0.1 \pi x) \sinh(0.1 \pi y) / \sinh(\pi)$$
(12)



Figure 4. Graphical comparison between numerical solution and exact solutions.

In Table 1 A, $\|.\|_0$ means the usual L^2 norm, $\|.\|_1$ is the usual semi H^1 norm, and obviously all norm are computed numerically subsequent to the mesh used¹⁴⁻¹⁶. The quantity $\|.\|_{\infty}$ is the discrete infinite norm L^{∞} that shows the highest of the absolute value of the known function at the nodes of the mesh. Where,

$$H^{1} = \frac{\sum_{i=1}^{N} \left| i \mathcal{U}_{i} - \overline{u}_{i} \right|}{N} \tag{18}$$

$$L^{2} = \sqrt{\frac{\sum_{i=1}^{N} \left(i \vartheta_{i} - \overline{u}_{i}\right)^{2}}{N}}$$
(19)

$$L^{\infty} = \max \sum_{i=1}^{N} \left| i \mathcal{U}_{i} - \overline{u}_{i} \right|$$
⁽²⁰⁾

Table 1. L^{∞} , L^2 and H^1 error of the Galerkin method solution

Nel	Size(h)	$\left\ \widetilde{u}_{i}-\overline{u}_{i}\right\ _{0}$	$\left\ \widetilde{u}_{i}-\overline{u}_{i}\right\ _{2}$	$\ \widetilde{u}_i - \overline{u}_i\ _{\infty}$
4	1.23	7.1264	7.1264	1.0349
8	0.625	7.3429	1.1313	0.27015
16	0.3125	7.2243	0.55609	0.069297
32	0.15625	7.979	0.27469	0.017373
64	0.099125	7.036	0.13639	0.0043522
128	0.039063	6.9972	0.067954	0.009996
256	0.019531	6.9799	0.033931	0.00027255

Generated general regression form Error = $a h^b$

$$E = ah^b \tag{21}$$

where, h is the elements size, and a and b are two constants to be determine from the actual values of the interpolation and solution errors for each case. For error analysis with linear regression we can see that the data in Table 1. Obey.

$$\|u_{i}^{\prime} - \overline{u}_{i}\|_{0} \approx 0.2269 \ h^{1.9902}$$
 (22)

$$\|u_{i0}^{0} - \overline{u}_{i}\|_{2} \approx 0.08310 \ h^{1.0419},$$
 (23)

$$\|u_{i}^{\prime} - \overline{u}_{i}\|_{\infty} \approx 0.4342 \ h^{1.9381}$$
 (24)

which indicates that the result u_h converge to the exact solution with the convergence rates of almost O(h²) in the L^2 norm, O(h) in the H^1 norm and O(h²) in the discrete perpetuity norm. Figures 5(a) and 5(b) illustrate that inaccuracy results for the growing number of the elements and decreasing size of the elements.



Figure 5. Error analysis for (a) increasing of the elements number, (b) decreasing of the elements size.



Figure 6. Surface plot for 2D heat conduction PDEs.





Figure 7. Contour plot using by (a) present method and (b) Ansys.

Figure 6 presents a surface plot for the 2D heat conduction. Figure 7(a) depicting the contour plot of the present solution and Figure 7(b) depicting the contour plot of ANSYS solutions show that the heat temperature distribution in the whole region. It is clear from the contour plot that results obtained by the present work are very close to the ANSYS solution.

5. Conclusion

The Galerkin finite element method using three-nodes triangular element models constructed in the present

work shows a convergence result by increasing the number of elements or decreasing the size of elements. The accuracy of the present model is demonstrated by comparison with analytical solution as well as ANSYS result.

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7. References

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